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The bifurcation and secondary bifurcation of capillary-gravity waves in the presence of symmetry

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THE BIFURCATION AND SECONDARY BIFURCATION OF CAPILLARY-GRAVITY WAVES IN THE PRESENCE OF SYMMETRY

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CONTENTS

S U M M A R Y

This thesis is concerned with the existence, multiplicity and properties of two-dimensional, symmetric, periodic waves which arise as the free surface of an ideal fluid which is in steady motion in a deep horizontal channel under the forces of gravity and surface tension. The parameters in this problem are the phase speed and the surface tension. The surface tension will first of all be regarded as fixed. We shall then see that there are certain values of the phase speed, known as eigenspeeds, at which solution curves of small amplitude capillary-gravity waves may bifurcate from the flat free surface. Also, along these curves a secondary bifurcation, corresponding to an n-fold increase in period, may take place. Thus for certain values of the phase speed close to an eigenspeed a number of different water-waves are possible. These waves may differ from each other both in respect of their minimal periods and their symmetries: some are symmetric only about crests, some only about troughs and some about both crests and troughs. Then we shall study the effect on the solution set of perturbing the surface tension. This can have a dramatic effect: bifurcation and secondary bifurcation points can exchange rôles or secondary bifurcations may appear or disappear.

The work is ordered as follows: Chapter I contains an outline of the problem and the main results as well as a brief literature survey. Chapter II contains a more detailed description of the problem and of how it is transformed into a non-linear integral equation (N) for θ, the angle between the free surface of the wave and the horizontal. Also various symmetries and invariant subspaces which

naturally occur in the problem are described. Chapter III contains various technical results regarding function spaces. In Chapter IV we describe the various techniques of bifurcation theory by which the problem is studied, most importantly the method of Lyapunov-Schmidt whereby the infinite dimensional problem may be reduced to one of finite dimensions. This procedure is then carried out and two polynomial equations in four unknowns result. Then in Chapter V these equations are analysed and bifurcation diagrams for (N) are obtained. Finally in Chapter VI these are interpreted in the context of the underlying hydrodynamical problem.

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C H A P T E R I

INTRODUCTION

1.1 The Problem and the Main Results

Consider the possible steady, two-dimensional irrotational motion of an ideal fluid which is contained in a channel of infinite horizontal extent whose depth is also infinite. The forces on the fluid are gravity (which is assumed constant throughout) and surface tension. Clearly one such motion is uniform horizontal laminar flow with flat free surface. In this thesis we investigate the existence of flows for which the free surface consists of a permanent wave train consisting of periodic and symmetric waves, the flow below being asymptotic to uniform horizontal flow with constant speed c at infinite depth. There are two parameters present in this problem: c, the phase speed and T, the surface tension. For fixed values of T we will regard c as the bifurcation parameter and consider for what values of c small amplitude solutions may occur when the free surface is close to being horizontal and the motion is almost uniform. It is found that for certain values of the phase speed, bifurcation of (possibly more than one) periodic wave trains may occur and that as the phase speed is altered these waves may undergo a secondary bifurcation corresponding to an n-fold increase in their period. Thus for certain values of c a number of distinct wave configurations are possible. It was mentioned that we only seek symmetric waves and indeed we shall find that some of these distinct wave trains may have the same period but differ in the type of symmetry they enjoy: some are symmetric only about crests, some only about troughs and some about both crests and troughs. Having thus found the bifurcation diagrams for fixed values of the surface tension T we seek the effect on them of perturbing T.

It will be seen that for certain critical values of T this perturbation can have a dramatic effect on the bifurcation diagrams: curves of solutions may appear or disappear and secondary bifurcation points may move from one curve to another. So far our remarks have been confined to small amplitude waves and the bulk of this thesis will be concerned with them. However there are some results regarding the global theory and it is shown, for instance, that in certain circumstances, the waves which bifurcate from the flat free surface are part of a global continuum of solutions which can only satisfy a number of possibilities.

If we give a brief description of how the problem is approached then we can be more specific about the results that are obtained, and what factors determine these results. The method of studying the problem will be via a new non-linear integral equation formulation for θ, the angle between the wave profile and the horizontal. The derivation of this equation is given in detail in Chapter II but it is (apart from minor changes in the parameters used):

$$T\theta(s) = \left(\int_0^\pi \exp\ \tau(w)\ dw\right)^{-1} \left\{c^2\int_0^s\int_0^\pi \sinh(\tau(x)-\tau(w))\ dw\ dx\right.$$
$$\left. + g\int_0^s\int_0^\pi\int_y^x \exp(\tau(x)+\tau(y)+\tau(w))\sin\ \theta(w)\ dw\ dy\ dx\right\} \qquad (\tilde{N}).$$

Here g is the acceleration due to gravity, which we take to be a constant throughout. Also, $\theta \in X_1$ which we define as the Banach space of 2π-periodic, continuously differentiable, odd functions; and $-\tau$ is the L_2 conjugate of θ. (We assume, without loss of generality, that all the waves have period 2π, though this need not be the minimal period.)

First of all, suppose T is held fixed. It is obvious that $\{(c,0) : c \in \mathbb{R}\}$ is a line of solutions of $(\tilde{N})$, corresponding to uniform horizontal laminar flow. These are known as the trivial solutions. It is well known from the classical theory of bifurcation that a necessary condition for $(c^*,0)$ to be a bifurcation point is that there should exist a $\theta^* \neq 0$ such that (c^*,θ^*) is a solution of equation $(\tilde{N})$ linearised about $\theta = 0$; such values of c will be termed eigenspeeds. It turns out that whatever the value of T, then the spectrum of the linearised problem consists of a countable set $\{c_k\}_{k\in\mathbb{N}}$ and in general the eigenspace corresponding to c_k is one dimensional and is spanned by sin ks. However, if T has a value T_{NM} such that $T_{NM} = g(NM)^{-1}$, where M and N are distinct natural numbers, it turns out that the spectrum contains an eigenspeed c_{NM} whose eigenspace is two dimensional, being spanned by sin Ns and sin Ms. Such an eigenspeed will be termed a critical eigenspeed. If an eigenspeed is not a critical eigenspeed it will be termed as simple eigenspeed.

We can now make a more precise statement of the problems which we shall study.

(I) The first problem is that of the existence of a solution curve which bifurcates from the horizontal flow at a simple eigenspeed c_k. This question has a relatively simple answer: it is a consequence of the classical "theory of bifurcation from a simple eigenvalue" that a family of solutions consisting of waves whose minimal period is $2\pi/k$ bifurcates from the horizontal flow at c_k. These solutions are unique in a neighbourhood of $(c_k,0)$; outside this neighbourhood they form part of a globally defined set of solutions.

(II) The second problem is essentially to determine the solution set of $(\tilde{N})$ in a neighbourhood of $(T_{NM}, c_{NM}, 0)$. The answer to this question is more complicated and more interesting and we shall consider it in stages.

(a) First suppose T is fixed at T_{NM}. Then the answer is that a multiplicity of solutions bifurcates from the trivial flow as c passes through its critical value c_{NM}.

Precisely how many, and the exact nature of the bifurcation, depends on the values of M and N. For instance, suppose that $M > N$ and $M \neq kN$ for all $k \in \mathbb{N}$. Then if $M > 2N$ bifurcation of two branches of capillary-gravity waves occurs from c_{NM}. One branch bifurcates sub-critically and consists of waves whose minimal period is $2\pi/N$ and the other branch bifurcates super-critically and consists of waves whose period is $2\pi/M$. All these waves exhibit both crest and trough symmetry.

If $M < 2N$ then there are two branches of solutions as above, but this time they both bifurcate sub-critically. In addition there are two more sub-critical bifurcations of waves whose minimal period is $2\pi/K$ (where K is the highest common factor of M and N). The waves on these additional solution branches may all exhibit both crest and trough symmetry or one set may enjoy only crest symmetry and the other only trough. Which alternative actually occurs depends on the values of M and N.

Full details of these results and also of the bifurcation which occurs when $M = kN$, $k \in \mathbb{N}$ are given in Chapters V and VI.

(b) Now we ask what happens to these waves if the surface tension is perturbed so $T \sim T_{NM}$. Again there are a number of different cases but they all share certain common features.

The critical eigenspeed c_{NM} is perturbed into two simple eigenspeeds c_M and c_N with corresponding eigenfunctions $\sin Ms$ and $\sin Ns$ respectively. Then by (I) it is known that a family of waves whose minimal period is $2\pi/M$ and $2\pi/N$ bifurcates from c_M and c_N respectively.

However, it will be shown that either of these branches may contain a secondary bifurcation point but this depends both on the values of M and N and on whether $T > T_{NM}$ or $T < T_{NM}$. To take an example: suppose that $M \neq kN$ and $M > 2N$. Then if $T > T_{NM}$ both solution branches contain a value of the phase speed at which the waves of minimal period $2\pi/N$ (or $2\pi/M$) bifurcate into waves of minimal period $2\pi/K$. Further, the secondary bifurcation curves join up to form a loop connecting the two secondary bifurcation points and the two branches of this loop each consist of a different set of capillary-gravity waves.

On the other hand if $T < T_{NM}$ then secondary bifurcation does not occur on either primary branch of solutions.

The results corresponding to the other cases are given in Chapter V.

1.2 The Method

In this section we provide a brief exposition of how equation $(\tilde{N})$ is arrived at and also of the techniques used to investigate its solutions.

Because of our periodicity and symmetry assumptions, one period of the motion must occupy the region S in the complex z-plane bounded by the lines $x = \pm\pi$ and the curve $\{x+iH(x) : -\pi \le x \le \pi\}$, where H is a smooth, even 2π-periodic function which is "a priori" unknown. Since the motion is irrotational there exists a function ψ, called the stream function, which is related to the velocity of the flow at a point $z \in S$ by $(u(z), v(z)) = (-\psi_y(z), \psi_x(z))$. It will be shown in detail in Chapter II that as a result of the physical assumptions about the flow, ψ must satisfy the following conditions:-

$$\Delta\psi = 0 \quad \text{on } S,$$

$$\psi_x(\pm\pi+iy) = 0, \quad y < H(\pm\pi),$$

$$(\psi_y, -\psi_x) \to (c\ ,0) \quad \text{as } y \to -\infty,\ (x,y) \in S,$$

$$\psi(x+iH(x)) = 0, \quad x \in [-\pi, \pi],$$

$$\psi(x+iy) \to -\infty \quad \text{as } y \to -\infty, \quad (x,y) \in S,$$

$$\tfrac{1}{2}|\nabla\psi(x+iH(x))|^2 + gH(x) - T\left(\frac{H'(x)}{(1+H'(x)^2)^{\frac{1}{2}}}\right)' = \text{constant}, \quad x \in [-\pi,\pi].$$

(This last equation is a form of Bernoulli's condition, modified to take account of the effects of surface tension.)

Thus the question of the existence of capillary-gravity waves is reduced to that of the existence of solutions to a free boundary value problem. The idea now is to transform the region S onto the unit disc in the complex w-plane. This mapping takes the free surface onto the unit circle and by choosing the constant in Bernoulli's condition suitably it is ensured that if $\theta(s)$ is the slope of the wave profile at the point corresponding to e^{is}, $s \in [-\pi,\pi]$, then θ satisfies $(\tilde{N})$. Thus solutions of $(\tilde{N})$ correspond to solutions of the capillary-gravity wave problem.

It is necessary to choose a suitable Banach space for θ. Because of the symmetry and periodicity assumptions a natural space to choose is that of continuously differentiable, odd, 2π-periodic functions which, recall, we have denoted by X_1. However, if we define, for $n \in \mathbb{N}$, X_n to be the subspace of X_1 consisting of $2\pi/n$ periodic functions then it turns out that these spaces are invariant under the action of the integral operator defined by the right-hand side of $(\tilde{N})$ and thus it is possible to find solutions $\theta \in X_n$, $n > 1$. Such solutions correspond to waves of period $2\pi/n$. The existence of these invariant subspaces is a reflection of symmetries which naturally occur in the physical problem. As we shall see, an awareness of these symmetries is essential for a full mathematical analysis of $(\tilde{N})$ and also to interpret solutions θ of $(\tilde{N})$ in the context of capillary-gravity waves.

To determine whether bifurcation occurs from an eigenspeed c, different methods are used, depending on whether c is a critical eigenspeed or not. If c is not a critical eigenspeed then the null space of the linearised equation is one dimensional and the existence of a global branch of solutions which bifurcates from (c, 0) follows without difficulty from the classical theory of bifurcation from a simple eigenvalue. (It might be worth noting that although c may be a critical eigenspeed in the context of X_1, it may nevertheless be a simple eigenspeed in the context of X_n for some $n > 1$.) If c is a critical eigenspeed, the null space of the linearised equation is two dimensional and the question of whether bifurcation takes place or not is much less straightforward. In this case, solutions of $(\hat{N})$ may be sought by the method of Lyapunov-Schmidt. This is a procedure whereby, using the Implicit Function Theorem, the infinite dimensional problem $(\hat{N})$ may be reduced to one of finite dimensions. Specifically it may be shown to be equivalent to two polynomial equations in four unknowns. The series expansion of these polynomials may in principle be computed to any degree of accuracy but due to the rapidly increasing labour involved only terms up to cubic order are evaluated. These polynomials contain no quadratic terms unless $M = 2N$ so this is one reason why the bifurcation diagrams differ according to the values of M and N.

The next stage is to analyse the bifurcation equations. Since the equations can only be determined up to cubic order we have to infer results about the full equations from the truncated equations. There are two basic tools in this analysis: one is the classical Implicit Function Theorem and the other is the Blowing-Up Lemma. The latter, which is stated precisely in Chapter V, is a tool from differential geometry which, roughly speaking, says that in a qualitative sense the solution set of the truncated equations in a neighbourhood of the origin is "the same" as the solution set of the full equations there.

Having found the solutions of the bifurcations equations and hence the solutions θ of $(\hat{N})$ it remains to interpret them in the context of the original hydrodynamical problem. Here we must make use of the symmetries inherent in the problem because, as we shall see, different functions θ may correspond to the same physical wave, but referred to a different origin. However by making use of these symmetries we may enumerate precisely the number of distinct waves there are for each value of the phase speed near an eigenspeed and also determine what differences there are between them in respect of minimal period and crest and trough symmetry.

1.3 Related Work

Studies have been made of both the physical and analytical aspects of the problem discussed here. The definitive work on bifurcation from a simple eigenvalue is that of Crandall & Rabinowitz (1970) and Rabinowitz (1970). In the first paper the Implicit Function Theorem is used to prove that a curve of solutions always bifurcates from a simple eigenvalue and that this curve is unique in a neighbourhood of the bifurcation point. In the second paper the Leray-Schauder degree theory is used to prove that this branch is part of a global continuum of solutions. Bifurcation problems of the type in which a double eigenvalue splits, under perturbation, into two simple eigenvalues have been the subject of a series of papers by Golubitsky & Schaeffer (1979 a,b,c) and Shearer (1978, 1980). They obtain the existence of secondary bifurcation points as a result of the invariance of the bifurcation equations under the actions of certain symmetry groups. The Lyapunov-Schmidt procedure and the analysis of the resulting bifurcation equations is described in detail in the book by Chow & Hale (1982). Our bifurcation diagrams are very similar to those obtained by all these authors. Indeed the capillary-gravity wave problem is of particular interest in this context since it is a single physical situation in which many different bifurcation phenomena may be observed.

The most extensive study of capillary-gravity waves is that undertaken by Chen & Saffman (1979). They develop a "weakly non-linear" theory by employing a formal power series expansion approach.

The results thus obtained appear to confirm those which are presented here but these authors do not always seem to fully appreciate when different mathematical solutions correspond to the same wave. In a later paper (1980) the same authors employ numerical techniques to extend the results to waves of finite amplitude. Reeder & Shinbrot (1981) made a rigorous analysis of the case M = 2, N = 1. The waves which arise in this situation are also known as "Wilton ripples". Their results agree with those of Chen & Saffman and those presented here. However, as we shall see, in that particular case the calculations are much simpler and the result is not typical.

Nekrasov (1920), Levi-Civita (1925), Milne-Thomson (1968) and Amick & Toland (1981) have all made a study of the periodic water-wave problem in which surface tension is neglected. They all employ a similar method to ours in that a hodograph transformation is used to map a region in the physical plane occupying one wavelength onto the unit disc and thereby a non-linear integral equation for θ, the wave slope, is obtained. (This equation has various equivalent forms some of which include the L_2-conjugate of θ.) Roughly speaking, these authors obtain a Neumann problem for θ, and thus their integral equations all contain a Green's function. However, as will become clear, we adopt a different approach and $(\tilde{N})$ contains no Green's function. Hence in the case of zero surface tension $(\tilde{N})$ does not reduce to the form given in any of this previous work. Indeed, so far as we know $(\tilde{N})$ is new in the present context.

Beale (1979) studied the problem of capillary-gravity waves along a horizontal channel whose depth is finite. His approach was more akin to that of previous authors and he obtained results valid for waves with small amplitude and large wavelength. His results can be deduced from those given here in the context of bifurcation from a simple eigenvalue.

Sekerzh-zen'kovich (1963, 1968, 1970) has considered a number of aspects of the capillary-gravity wave problem. In (1968) he considers the pure capillary-gravity wave problem while in (1963) and (1970) respectively he considers capillary-gravity waves over a wavy bottom or induced by a periodic pressure distribution over the free surface, respectively. In each case he obtains an integral equation for θ, the wave slope. He then proceeds to find small amplitude solutions by means of a formal power series expansion.

An extensive numerical study of the properties of capillary-gravity waves has been made by Hogan (1979, 1980, 1981). When the depth is finite, Hunter & Vanden-Broeck (1983) investigated numerically the convergence of finite amplitude capillary-gravity waves to solitary waves in the long wave limit. Their results indicate the existence of solitary waves of depression which are the long wave limit of periodic waves. They neither proved nor disproved the existence of solitary waves of elevation although they showed that if such waves do exist then they cannot be the long wave limit of periodic waves of elevation. Finally Amick and Kirchgässner have proved rigorously that when the values of the surface tension belong in a certain range, solitary capillary-gravity waves do indeed exist.

CHAPTER II

THE WATER-WAVE PROBLEM

In this chapter we shall first give a detailed description of the physical situation which we call the capillary-gravity wave problem and then show how it can be formulated mathematically as a free boundary value problem. Then we shall give the details of how it can be transformed into the non-linear integral equation (N) for θ, the wave slope. We shall then describe how the various symmetries occuring naturally in the physical problem are reflected in the mathematical properties of the integral operator on the right-hand side of (N) and show how different solutions θ of (N) may correspond to the same capillary-gravity wave.

2.1 The Physical Problem

Under consideration are steady, periodic, two-dimensional travelling waves which arise as the free surface of an inviscid, incompressible fluid of constant density which is contained in a channel of infinite horizontal extent whose depth is also infinite. The forces of gravity and surface tension act on the fluid and on the free surface the pressure is atmospheric, which we take to be a constant. The wave profile can be brought to rest by superimposing a uniform horizontal velocity on the whole system in the opposite direction to that in which the waves are travelling. We shall only consider waves which are symmetric around a fixed vertical line which will be taken to be the y-axis of an x-y co-ordinate system where gravity acts in the negative y-direction. We take the origin to be on the free surface so by our symmetry assumption it is situated at a crest or a trough. At infinite depths, the motion is horizontal with velocity c in the negative x-direction. (The quantity c is known as the <u>phase speed</u>.) Further, the waves are assumed periodic with period λ where λ is an arbitrary but <u>fixed</u> positive number. It is no loss of generality then to scale the x-co-ordinate so that the period of the motion becomes 2π. (Note that 2π need not be the minimal period.) Therefore a cross-section of the flow perpendicular to the wave crests may be identified with the region in the complex z-plane $\{y < H(x) : x \in \mathbb{R}\}$ where $H(x)$ is a smooth, even, 2π-periodic function which satisfies $H(0) = 0$. Since the wave profile has a crest or trough at the origin, H must have a local maximum or minimum there. (Of course H is not known "a-priori".) The curve $\{x + iH(x) : x \in \mathbb{R}\}$ is called the <u>free surface</u>.

At any point z within the fluid let the velocity be $(u(z),v(z))$. Then by periodicity both u and v have period 2π. Further, it follows from the fact that the motion is symmetric about $y = 0$ that

$$u(x+iy) = u(-x+iy) \quad , \tag{2.1(a)}$$

$$v(x+iy) = -v(-x+iy) \quad . \tag{2.1(b)}$$

Combining (2.1(b)) with periodicity we have that

$$v(\pm\pi+iy) = 0 \quad . \tag{2.2}$$

The free surface is a streamline and hence Bernoulli's theorem applied there yields that

$$p+\tfrac{1}{2}(u^2(x+iH(x))+v^2(x+iH(x)))+gH(x) = \text{const}, \; x \in \mathbb{R}. \tag{2.3}$$

Here p is the pressure within the fluid and g is the acceleration due to gravity. (The density is assumed to be unity.) Since we are not neglecting surface tension there is a pressure difference across the free surface which is given by

$$p_o - p = \frac{T}{R} , \tag{2.4}$$

where p_o is the constant atmospheric pressure, R is the radius of curvature of the free surface and T, the surface tension, is a positive constant. On the free surface $y = H(x)$ and so

$$\frac{1}{R} = \frac{d}{dx}\left(\frac{H'(x)}{(1+H'(x)^2)^{\frac{1}{2}}}\right) = \frac{H''(x)}{(1+H'(x)^2)^{3/2}} . \tag{2.5}$$

Thus (2.3) becomes

$$\tfrac{1}{2}(u^2(x+iH(x))+v^2(x+iH(x)))+gH(x)-T\frac{H''(x)}{(1+H'(x)^2)^{3/2}} = \text{const}, \ x \in \mathbb{R} . \tag{2.6}$$

2.2 The Mathematical Formulation

In this section it will be shown how the physical problem of §2.1 can be reformulated as an elliptic free boundary value problem. First observe that the fluid in one period of the flow occupies the region S in the complex z-plane bounded by the lines $x = \pm\pi$ and the curve $\Gamma = \{x+iH(x) : x \in (-\pi,\pi)\}$. Since the fluid is incompressible and the flow irrotational there exists a function analytic in S, the complex potential, $w = \phi + i\psi$ which is related to the velocity $(u(z),v(z))$ of the flow at a point $z \in S$ by the expression

$$u(z) - iv(z) = -\frac{dw}{dz} = -\phi_x(z) + i\phi_y(z) = -\psi_y(z) - i\psi_x(z). \tag{2.7}$$

(Since the flow is 2π-periodic, w is 2π-periodic also.)

ϕ is known as the velocity potential and ψ as the stream function. It follows from (2.1(b)) that

$$\psi_x(x+iy) = -\psi_x(-x+iy), \tag{2.8}$$

so

$$\psi(x+iy) = \psi(-x+iy) , \tag{2.9}$$

and (2.2) implies that

$$\psi_x(\pm\pi+iy) = 0, \; y < H(\pm\pi). \tag{2.10}$$

Since at infinite depths the motion is horizontal with velocity - c in the positive x-direction we have

$$(\psi_y, -\psi_x) \to (c, 0) \qquad \text{as } y \to -\infty. \tag{2.11}$$

The free surface Γ is a streamline and so ψ is constant along it. Thus it is no loss of generality to assume

$$\psi(z) = 0, \; z \in \Gamma. \tag{2.12}$$

At infinite depths the flow is horizontal so

$$\psi(z) \to -\infty \qquad \text{as } y \to -\infty. \tag{2.13}$$

The final condition is that of Bernoulli (2.6). Since $u^2(z) + v^2(z) = |\nabla\psi(z)|^2$ this becomes

$$\tfrac{1}{2}|\nabla\psi(x+iH(x))|^2 + gH(x) - T\frac{H''(x)}{(1+H'(x)^2)^{3/2}} = \text{const.} \quad x \in [-\pi,\pi]. \tag{2.14}$$

Thus the capillary-gravity wave problem with surface tension T and phase speed c has been formulated mathematically as a free boundary value problem. By a solution to this problem (for given T and c) we mean that there exist two functions H(x) and w(z). The first, H(x), defines a region S and the second, w(z) is analytic in S. Furthermore, H(x) and w(z) must satisfy the conditions (2.8) to (2.14).

2.3 The Integral Equation

In this section we will show how the existence question to the capillary-gravity wave problem may be studied via a Volterra type integral equation. First, though, we need some results from the theory of Fourier series. To avoid unnecessary complications we consider only smooth (C^∞) functions, later, in Chapter III we shall consider function spaces in more detail.

If u is a smooth, 2π-periodic, odd function, the series

$$\sum_{k=1}^{\infty} a_k \sin kt \tag{2.15}$$

where

$$a_k = \frac{2}{\pi} \int_0^\pi u(t) \sin kt \, dt \tag{2.16}$$

is called the Fourier series of u.

The function v defined by

$$v(t) = - \sum_{k=1}^{\infty} a_k \cos kt \tag{2.17}$$

is called the conjugate of u.

Since u is smooth it follows from Privalov's theorem: p.121 of Zygmund (1959), also p.99 of Bary (1964), that v is smooth also.

The next theorem is a precise statement of how solutions of a certain integral equation correspond to capillary-gravity waves.

THEOREM 2.1. <u>Suppose</u> θ <u>is a smooth, odd,</u> 2π-<u>periodic function which satisfies</u> $|\theta(s)| < \pi/2$ <u>for all</u> s <u>and</u> $\theta(\pi) = 0$. <u>Let</u> $-\tau$ <u>be the conjugate of</u> θ. <u>Then if for some positive numbers</u> ν <u>and</u> γ, <u>the equation</u>

$$\gamma\theta(s) = \left(\int_0^\pi \exp \tau(w)\, dw\right)^{-1} \left\{\nu\int_0^s\int_0^\pi \sinh(\tau(x)-\tau(w))\, dw\, dx \right.$$
$$\left. + \int_0^s\int_0^\pi\int_y^x \exp(\tau(x)+\tau(y)+\tau(w))\sin\theta(w)\, dw\, dy\, dx\right\},\quad s \in [-\pi,\pi]\,(N)$$

<u>is satisfied, there exists a solution to the capillary-gravity wave problem with</u> $T = g\gamma$ <u>and</u> $c^2 = g\nu$. <u>Moreover there exists a parameterization of the free surface</u> Γ, $\{(x(t),y(t)) : t \in [-\pi,\pi]\}$ <u>say, such that</u> $\theta(t)$ <u>gives the angle between the free surface and the horizontal at the point</u> $(x(t),y(t))$.

<u>Proof</u>. Differentiate (N) and rearrange to obtain

$$\gamma\theta'(s)\int_0^\pi e^{\tau(w)}\, dw - \left(\frac{\nu}{2}\right)\left(e^{\tau(s)}\int_0^\pi e^{-\tau(w)}\, dw - e^{-\tau(s)}\int_0^\pi e^{\tau(w)}\, dw\right)$$
$$- e^{\tau(s)}\left(\int_0^\pi e^{\tau(y)}\, dy\right)\left(\int_0^s e^{\tau(w)}\sin\theta(w)\, dw\right)$$
$$= -e^{\tau(s)}\left(\int_0^\pi e^{\tau(y)}\int_0^y e^{\tau(w)}\sin\theta(w)\, dw\, dy\right), \tag{2.18}$$

whence

$$\gamma e^{-\tau(s)}\theta'(s) + \left(\frac{\nu}{2}\right)e^{-2\tau(s)} - \int_0^s e^{\tau(w)}\sin\theta(w)\, dw = \text{const.}\quad s \in [-\pi,\pi]. \tag{2.19}$$

Now let

$$f(\xi) = \sum_{k=1}^{\infty} a_k \xi^k \qquad |\xi| \le 1$$

in the complex ξ plane where $a_k = \frac{2}{\pi}\int_0^{\pi} \theta(t)\sin kt\, dt, \quad k \ge 1$.
Then if $f(\xi) = \tilde{\tau}(\xi) + i\tilde{\theta}(\xi)$ where $\tilde{\tau}$ and $\tilde{\theta}$ are real valued it follows that $\tilde{\theta}(e^{it}) = \theta(t)$ and $\tilde{\tau}(e^{it}) = \tau(t)$.

Let $\mathcal{D} = \{\xi : |\xi| \le 1, \xi \notin [-1,0]\}$ and define a function m in $\mathcal{D}$ by

$$m(\xi) = i\int_1^{\xi} \frac{\exp(f(\rho))}{\rho}\, d\rho \tag{2.20}$$

where the integration is carried out along contours in $\mathcal{D}$. Now since $|\theta(s)| < \pi/2$, $s \in [-\pi,\pi]$, it follows that $|\tilde{\theta}(\xi)| < \pi/2$, $|\xi| \le 1$. Using these facts we can deduce m is an injective function.

For suppose there exist $\xi_1, \xi_2 \in \mathcal{D}$, $(\xi_1 \neq \xi_2)$ such that $m(\xi_1) = m(\xi_2)$. Then if we transform m, via the one-to-one transformation $\rho = e^{-i\zeta}$, into an integral on the region $\{\zeta : -\pi < \text{Re}\ \zeta < \pi, \text{Im}\ \zeta \le 0\}$, we obtain

$$m(\xi_j) = \int_0^{i \ln \xi_j} \exp(f(e^{-i\zeta}))\, d\zeta \qquad j = 1, 2.$$

Thus $m(\xi_1) - m(\xi_2) = 0$ and on taking real parts we have

$$\int_{i \ln \xi_1}^{i \ln \xi_2} \exp \tilde{\tau}(e^{-i\zeta})\cos \tilde{\theta}(e^{-i\zeta})\, d\zeta = 0,$$

which is a contradiction since $|\tilde{\theta}| < \pi/2$.

Moreover

$$m'(\xi) = \frac{i\ \exp(f(\xi))}{\xi} \neq 0,$$

and it therefore follows that m is a conformal mapping from $\mathcal{D}$ onto some region S of the complex z-plane and that m is invertible on S. We wish to determine S.

Observe first that an integration of $\frac{i\ \exp f(\rho)}{\rho}$ around the contour

$$C_\varepsilon = \{e^{is} : s \in (-\pi,\pi)\} \cup \{r + i0 : r \in [-1,-\varepsilon]\}$$
$$\cup \{\varepsilon e^{-is} : s \in (-\pi,\pi)\} \cup \{-r-i0 : r \in [\varepsilon,1]\}$$

for arbitrary $\varepsilon > 0$ yields that

$$-\int_{-\pi}^{\pi} \exp f(e^{it})\ dt + \int_{-1}^{-\varepsilon} \frac{\exp f(re^{+i\pi})}{r}\ dr$$

$$+\int_{-\pi}^{\pi} \exp f(\varepsilon e^{-it})\ dt + \int_{-\varepsilon}^{-1} \frac{\exp f(re^{-i\pi})}{r} dr = 0.$$

The second and fourth integrals cancel out since $\tilde{\tau}(re^{i\pi}) = \tilde{\tau}(re^{-i\pi})$ and $\tilde{\theta}(re^{\pm i\pi}) = 0$. Then letting $\varepsilon \to 0$ and using the fact that $f(0) = 0$ we obtain that

$$2\int_{0}^{\pi} \exp \tau(t)\cos \theta(t)\ dt$$

$$= \int_{-\pi}^{\pi} \exp \tau(t)(\cos \theta(t) + i \sin \theta(t))\ dt \qquad \text{(since } \theta \text{ is odd)}$$

$$= \int_{-\pi}^{\pi} \exp \tilde{\tau}(e^{it})(\cos \tilde{\theta}(e^{it}) + i \sin \tilde{\theta}(e^{it}))\ dt$$

$$= \int_{-\pi}^{\pi} \exp f(e^{it})\ dt = 2\pi.$$

Also if $\varepsilon e^{i\theta} \in \mathcal{D}$ then

$$m(\varepsilon e^{i\theta}) = i\int_1^\varepsilon \frac{\exp f(r)}{r}\,dr + \int_0^\theta \exp f(\varepsilon e^{it})\,dt$$

$$= -i\int_\varepsilon^1 \frac{\exp \tilde{\tau}(r)}{r}\,dr + \int_0^\theta \exp f(\varepsilon e^{it})\,dt$$

(since $\tilde{\theta}(r) = 0$ when r is real).

Then letting $\varepsilon \to 0$ we see that

$$\text{Im } m(\xi) \to -\infty \qquad \text{as } \xi \to 0 \text{ in } \mathcal{D}.$$

So it is clear that the region S is

$$S = \{z = x + iy : -\pi < x < \pi,\ y < H(x)\}$$

where

$$(x, H(x)) = -\left(\int_0^t e^{\tau(s)}\cos\theta(s)\,ds,\ \int_0^t e^{\tau(s)}\sin\theta(s)\,ds\right)\ t\in[-\pi,\pi]. \tag{2.21}$$

Hence $H'(x) = \tan\theta(t)$ where x is related to t by (2.21), and the free surface Γ is given parametrically by

$$\Gamma = \left(-\int_0^t e^{\tau(s)}\cos\theta(s)\,ds,\ -\int_0^t e^{\tau(s)}\sin\theta(s)\,ds\right). \tag{2.22}$$

Since m is invertible on S we may define a complex potential on S by writing

$$w(z) = \phi(z) + i\psi(z) = ic\,\ln(m^{-1}(z)). \tag{2.23}$$

Here ln denotes the usual branch of the logarithm which is defined everywhere except on the negative real axis. It remains to verify (2.8) to (2.14). First note that the velocity field in S generated by w is

$$- \frac{dw}{dz} = u(z) - iv(z) = -\psi_y(z) - i\psi_x(z)$$

$$= \frac{-ic}{m^{-1}(z)m'(m^{-1}(z))}$$

$$= -c \exp(-f(m^{-1}(z)))$$

$$= -c \exp(-\tilde{\tau}(m^{-1}(z)))(\cos \tilde{\theta}(m^{-1}(z)) - i \sin \tilde{\theta}(m^{-1}(z))). \qquad (2.24)$$

Hence at any point $z \in S$, $\tilde{\theta}(m^{-1}(z))$ gives the angle of the velocity vector to the horizontal and $c \exp(-\tilde{\tau}(m^{-1}(z)))$ gives the speed of the flow. Since θ is an odd function and τ is even it follows that $\tilde{\theta}(\bar{\xi}) = -\tilde{\theta}(\xi)$ and $\tilde{\tau}(\bar{\xi}) = \tilde{\tau}(\xi)$. Hence $\overline{m(\xi)} = -m(\bar{\xi})$ and so (2.8) and (2.9) are satisfied. If $\pm\pi + iy \in \bar{S}$ (the closure of S) then $m(re^{\pm i\pi}) = \pm\pi+iy$ for some $r \in (0,1]$. Thus $ic \ln(m^{-1}(\pm\pi+iy)) = c(\pm\pi+i \ln r)$ and so (2.10) is satisfied. Since $m^{-1}(z) \to 0$ as $|z| \to \infty$ in S and $f(0) = \tilde{\tau}(0) + i\tilde{\theta}(0) = 0$ it is clear that

$$u(z) - iv(z) \to -c + i0 \qquad \text{as } |z| \to \infty \quad \text{in S}$$

and so (2.11) and (2.13) hold.
Also $m^{-1}(\Gamma) = e^{it}$, $-\pi < t < \pi$ and so (2.12) holds.

It remains to verify (2.14). However

$$H'(x) = \tan\theta(t) \text{ and } H''(x) = -e^{-\tau(t)}\cos^{-3}\theta(t)\theta'(t) \tag{2.25}$$

if the point $(x,H(x)) \in \Gamma$ is given parametrically by (2.22). Now substituting (2.21) and (2.25) into the left-hand side of (2.14) and using the fact that

$$u^2(z) + v^2(z) = c^2\exp(-2\tilde{\tau}(m^{-1}(z))) \ , \tag{2.26}$$

we obtain

$$\frac{1}{2}c^2e^{-2\tau(t)} - g\int_0^t e^{\tau(s)}\sin\theta(s)\,ds + Te^{-\tau(t)}\theta'(t) \ ,$$

which is constant by (2.19).

q.e.d.

2.4 Symmetries and Invariant Subspaces

A question now arises as to which waves the solutions of (N) correspond. The point is that different solutions θ of (N) may represent the same capillary-gravity wave. The results contained in this section will enable us to identify different solutions of (N) as the same wave referred to a different origin. First we need some definitions. For any $N \in \mathbb{N}$ let

$$C_N^\infty = \{u \in C^\infty(\mathbb{R}) : u(t) = -u(-t) = u(t + \tfrac{2\pi}{N}) \ ; \ t \in \mathbb{R}\}, \tag{2.27}$$

so then $C_N^\infty \subseteq C_1^\infty$ for all $N \geq 1$.

Then for any $u \in C_1^\infty$ and $N \in \mathbb{N}$ define S_N by

$$(S_N u)(t) = u(t + \tfrac{\pi}{N}). \tag{2.28}$$

Further let Cu denote the conjugate of u as defined in (2.17). So if $u \in C_N^\infty$

$$u = \sum_{m=1}^{\infty} a_m \sin mNs \tag{2.29}$$

and

$$Cu = -\sum_{m=1}^{\infty} a_m \cos mNs \tag{2.30}$$

where $\{a_k\}$ is the sequence of Fourier coefficients.

Then $S_N \circ Cu$

$$= -S_N \sum_{m=1}^{\infty} a_m \cos mNs$$

$$= -\sum_{m=1}^{\infty} (-1)^m a_m \cos mNs$$

$$= C \sum_{m=1}^{\infty} (-1)^m a_m \sin mNs$$

$$= CoS_N(\sum_{m=1}^{\infty} a_m \sin mNs)$$

$$= CoS_N u.$$

For any $\theta \in C_1^\infty$, let $\tau = -C\theta$ and let $G(\nu,\theta)(s)$ denote the right-hand side of (N). In the rest of this chapter γ will be regarded as a fixed positive number and by a solution of (N) we will mean a pair $(\nu,\theta) \in (0,\infty)\times C_1^\infty$ satisfying $\gamma\theta = G(\nu,\theta)$. We now prove some theorems concerning the mathematical properties of $G(\nu,\theta)$.

THEOREM 2.2. For any $N \geq 1$ and any $(\nu,\theta) \in (0,\infty)\times C_N^\infty$, the function $G(\nu,\theta)(s)$ is in C_N^∞.

Proof. If $\theta \in C_N^\infty$, then since θ is odd and τ is even around $t = 0$ we obtain that

$$\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw$$

$$= \int_{-y}^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw, \tag{2.32}$$

for all $x, y \in \mathbb{R}$.

Now $\theta(\frac{\pi}{N} + s) = \theta(s + \frac{\pi}{N} - \frac{2\pi}{N})$

$$= \theta(s - \frac{\pi}{N}) = -\theta(\frac{\pi}{N} - s)$$

and hence θ is odd about π/N; similarly τ is even about π/N. Then combining this with the fact that both functions are $2\pi/N$-periodic we find that

$$\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw$$

$$= \int_{\frac{2\pi}{N}-y}^{x} \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw. \tag{2.33}$$

Therefore combining (2.32) and (2.33) we have

$$\int_0^{\pi/N} \int_0^{\pi} \int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx$$

$$= \begin{cases} \left\{\int_0^{\pi/N}\int_0^{\pi/N} + \sum_{k=1}^{m}\int_0^{\pi/N}\int_{((2k-1)\pi)/N}^{((2k+1)\pi)/N}\right\} \int_y^x \exp(\tau(x)+\tau(y)+\tau(w))\sin\theta(w)\, dwdydx & \text{if } N = 2m+1 \\ \sum_{k=1}^{m} \int_0^{\pi/N} \int_{(2(k-1)\pi)/N}^{2k\pi/N} \int_y^x \exp(\tau(x)+\tau(y)+\tau(w))\sin\theta(w)\, dwdydx & \text{if } N = 2m \end{cases}$$

$$= \begin{cases} m\int_0^{\pi/N}\int_{\pi/N}^{3\pi/N} \int_y^x \exp(\tau(x)+\tau(y)+\tau(w))\sin\theta(w)\, dwdydx & \text{if } N = 2m+1 \\ m\int_0^{\pi/N}\int_0^{2\pi/N} \int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dwdydx & \text{if } N = 2m \end{cases}$$

$$= \quad m\int_0^{\pi/N} \int_0^{2\pi/N} \int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dwdydx$$

(these two steps follow since all functions are $2\pi/N$-periodic)

$$= \quad 2m\int_0^{\pi/N} \int_0^{\pi/N} \int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dwdydx$$

(since the integrand w.r.t. y is even around π/N)

$$= \quad 0 \quad . \tag{2.34}$$

Now a change of variable yields that

$$\int_0^s \int_0^\pi \int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx$$

$$= -\int_0^{-s}\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx$$

and $\displaystyle\int_0^s\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx$

$$= \int_0^{s+\frac{2\pi}{N}}\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx. \qquad (2.35)$$

Combining these last two results it follows that

$$\int_0^{\frac{\pi}{N}+s}\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx$$

$$= -\int_0^{\frac{\pi}{N}-s}\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin\theta(w)\, dw\, dy\, dx.$$

Similarly

$$\int_0^s\int_0^\pi \sinh(\tau(x) - \tau(w))dw\, dx = -\int_0^{-s}\int_0^\pi \sinh(\tau(x) - \tau(w))\, dw\, dx,$$

and

$$\int_0^s\int_0^\pi \sinh(\tau(x) - \tau(w))dw\, dx = \int_0^{s+\frac{2\pi}{N}}\int_0^\pi \sinh(\tau(x) - \tau(w))\, dw\, dx,$$

so $\displaystyle\int_0^{\frac{\pi}{N}+s}\int_0^\pi \sinh(\tau(x) - \tau(w))dw\, dx = -\int_0^{\frac{\pi}{N}-s}\int_0^\pi \sinh(\tau(x) - \tau(w))\, dw\, dx.$

These results show that for each $\nu > 0$ and $\theta \in C_N^\infty$, $G(\nu,\theta)(s)$ is odd, $2\pi/N$ periodic and odd about π/N.

q.e.d.

The next result gives more information about the solution set of $(\mathcal{N})$.

THEOREM 2.3. <u>Suppose that $(\nu,\theta) \in (0,\infty)\times C_N^\infty$ is a solution of equation $(\mathcal{N})$. Then $(\nu, S_N\theta) \in (0,\infty)\times C_N^\infty$ is also a solution of $(\mathcal{N})$.</u>

<u>Proof.</u> Let $(\nu,\theta) \in (0,\infty)\times C_N^\infty$ be a solution of equation $(\mathcal{N})$. Then let $S_N\theta = \hat{\theta}$ and $C(S_N\theta) = -\hat{\tau}$ and recall from (2.28) that $\hat{\tau}(t) = \tau(t + \frac{\pi}{N})$. Then

$$\int_0^s\int_0^\pi\int_y^x \exp(\hat{\tau}(x) + \hat{\tau}(y) + \hat{\tau}(w))\sin \hat{\theta}(w)\ dw\ dy\ dx$$

$$= \int_0^s\int_0^\pi \exp(\hat{\tau}(x) + \hat{\tau}(y))\int_y^x \exp \tau(w + \tfrac{\pi}{N})\sin \theta(w + \tfrac{\pi}{N})\ dw\ dy\ dx$$

$$= \int_{\pi/N}^{s+\frac{\pi}{N}}\int_{\pi/N}^{\pi(1+\frac{1}{N})} \int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin \theta(w)\ dw\ dy\ dx$$

$$= \int_{\pi/N}^{s+\frac{\pi}{N}}\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin \theta(w)\ dw\ dy\ dx$$

$$\left(\text{since } \int_0^{\pi/N} e^{\tau(y)}\int_y^x e^{\tau(w)}\sin \theta(w)\ dw\ dy = \int_\pi^{\pi+\frac{\pi}{N}} e^{\tau(y)}\int_y^x e^{\tau(w)}\sin \theta(w)\ dw\ dy\right)$$

$$= \int_0^{s+\frac{\pi}{N}}\int_0^\pi\int_y^x \exp(\tau(x) + \tau(y) + \tau(w))\sin \theta(w)\ dw\ dy\ dx.$$

Similarly

$$\int_0^s\int_0^\pi \sinh(\hat{\tau}(x) - \hat{\tau}(w))\,dw\,dx = \int_0^{s+\frac{\pi}{N}}\int_0^\pi \sinh(\tau(x) - \tau(w))\,dw\,dx.$$

Therefore $G(\nu, S_N\theta) = S_N G(\nu,\theta)$, and hence $(\nu,S_N\theta)$ is a solution of $(\mathcal{N})$ whenever $(\nu,\theta) \in (0,\infty)\times C_N^\infty$ is a solution of $(\mathcal{N})$.

q.e.d.

However as will now be shown the additional solutions provided by the last theorem do not enlarge the set of solutions to the physical problem.

THEOREM 2.4. <u>Let $\theta \in C_N^\infty$. If (ν,θ) and $(\nu, S_N\theta)$ are solutions to $(\mathcal{N})$ they both correspond to the same capillary-gravity wave.</u>

<u>Proof</u>. If $\theta = S_N\theta$ the theorem is trivial. Note that this means $\theta(s) = \theta(s + \frac{\pi}{N})$ whence $\theta \in C_{2N}^\infty$. Suppose then that $\theta \in C_N^\infty \backslash C_{2N}^\infty$. Then the free surface of the corresponding capillary-gravity wave is

$$(x(t),y(t)) = \left(-\int_0^t e^{\tau(s)}\cos\theta(s)\,ds,\ -\int_0^t e^{\tau(s)}\sin\theta(s)\,ds\right).$$

A change of variable yields

$$(x(t),y(t)) = \left(\int_t^{\pi/N} - \int_0^{\pi/N}\right)\left(e^{\tau(s)}\cos\theta(s),\ e^{\tau(s)}\sin\theta(s)\right) ds$$

$$= -\int_0^{\pi/N}\left(e^{\tau(s)}\cos\theta(s),\ e^{\tau(s)}\sin\theta(s)\right) ds$$

$$-\int_0^{t-\frac{\pi}{N}}\left(e^{\hat{\tau}(s)}\cos\hat{\theta}(s),\ e^{\hat{\tau}(s)}\sin\hat{\theta}(s)\right) ds, \quad \text{where } \hat{\theta} = S_N\theta.$$

Hence

$$(x(t + \frac{\pi}{N}),\ y(t + \frac{\pi}{N})) - (x(\frac{\pi}{N}),\ y(\frac{\pi}{N})) = (\hat{x}(t),\ \hat{y}(t)),$$

where $(\hat{x},\hat{y})$ is defined in terms of $\hat{\theta}$ and $\hat{\tau}$ by (2.22). Therefore θ and $\hat{\theta}$ correspond to the same capillary-gravity wave, even though they are distinct solutions of (N).

q.e.d.

CHAPTER III

PRELIMINARY RESULTS AND FUNCTION SPACES

In this chapter we prove some technical results about the equation (N). In Chapter II , for the sake of simplicity, it was always assumed that θ was smooth. In Theorem 3.6 we shall prove that under certain conditions any continuous solution of (N) must be a smooth function. First we need some definitions.

Suppose $(a,b) \subset \mathbb{R}$ is an interval (not necessarily bounded). Then let $L_p[a,b]$, $p \geq 1$ denote the Banach space of p^{th} power integrable "functions" on $[a,b]$, the norm being given by

$$||f||_{L_p[a,b]} = \left(\int_a^b |f|^p \, dx \right)^{1/p} . \tag{3.1}$$

If f is a real valued function on (a,b) we write $f \in C^m(a,b)$ if f and all its derivatives up to and including order m are continuous in (a,b). The Banach space $C^m[a,b]$ denotes those functions in $C^m(a,b)$ whose derivatives up to and including m have continuous extensions up to a and b, the norm being

$$||f||_{C^m[a,b]} = \sup_{\substack{x \in [a,b] \\ 0 \leq k \leq m}} |f^{(k)}(x)| . \tag{3.2}$$

($f \in C^\infty[a,b]$ means $f \in C^m[a,b]$ for all $m \geq 0$.)
If $f \in C^0[a,b]$ and

$$|f(x) - f(y)| \leq K|x - y|^\alpha \quad \forall \; x, y \in [a,b] \tag{3.3}$$

where K is a constant and α is a constant belonging to the interval

$(0,1)$, then f is said to be Hölder continuous on $[a,b]$ with exponent α and we write $f \in C^{0,\alpha}[a,b]$. This is a Banach space with norm

$$||f||_{C^{0,\alpha}[a,b]} = \sup_{\substack{x,y\in[a,b] \\ |x-y|<1}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} . \tag{3.4}$$

The Banach space $C^{m,\alpha}[a,b]$ consists of those functions in $C^{m}[a,b]$ for which $f^{(m)} \in C^{0,\alpha}[a,b]$; the norm is

$$||f||_{C^{m,\alpha}[a,b]} = ||f||_{C^{m}[a,b]} + ||f^{(m)}||_{C^{0,\alpha}[a,b]} . \tag{3.5}$$

We now require some rather more detailed results from the theory of Fourier series than appeared in Chapter II. Suppose u is an odd, continuous, 2π-periodic function. Then the expression

$$\sum_{k=1}^{\infty} a_k \sin ks \tag{3.6}$$

where

$$a_k = \frac{2}{\pi}\int_0^{\pi} u(t)\sin kt \, dt$$

is called the Fourier series of u.

Then Cu, the conjugate of u is defined to be the function

$$Cu(s) \sim -\sum_{k=1}^{\infty} a_k \cos ks . \tag{3.7}$$

It follows from Theorem 2.4, page 253 of Zygmund, Volume I (1959) that the right-hand side of (3.7) is a Fourier series, and therefore that Cu is well defined. However Cu may not be a continuous function even though u is. The following theorem, due to Privalov, provides a link between the properties of a function and its conjugate. A proof can be found on page 121 of Zygmund, Volume I (1959) or page 99 of Bary (1964).

THEOREM 3.1. _If_ u _is odd and_ $u \in C^{0,\alpha}[-\pi,\pi]$ _for some_ $\alpha \in (0,1)$ _then_ $Cu \in C^{0,\alpha}[-\pi,\pi]$.

Throughout this thesis the spaces X_N, $N \geq 1$ defined below, are the ones most frequently used. For $N \geq 1$ let

$$X_N = \{u \in C^1(\mathbb{R}) : u(t) = -u(-t) = u(t + \frac{2\pi}{N}) \; ; \; t \in \mathbb{R}\} \tag{3.8}$$

also, for $\alpha \in (0,1)$ define

$$Y_N^\alpha = \{u \in C^{0,\alpha}(\mathbb{R}) : u(t) = u(-t) = u(t + \frac{2\pi}{N}) \; ; \; t \in \mathbb{R}\} \; . \tag{3.9}$$

These are both Banach spaces with norms inherited from $C^1(\mathbb{R})$, $C^{0,\alpha}(\mathbb{R})$ respectively. Then it is a consequence of Privalov's theorem that C maps X_N boundedly into Y_N^α, $N \in \mathbb{N}$, $\alpha \in (0,1)$.

We can now prove our first main result.

THEOREM 3.2. _For any_ $N \geq 1$, _if_ $(\nu,\theta) \in (0,\infty)\times X_N$ _then_ $G(\nu,\theta) \in X_N$. _Moreover_ $G : (0,\infty)\times X_1 \to X_1$ _is real analytic_.

Proof. It follows from Privalov's theorem that $\tau \in C^{0,\alpha}(\mathbb{R})$ with exponent α for any $\alpha \in (0,1)$ and so $G(\nu,\theta) \in C^1$ (indeed its first derivative is Hölder continuous with exponent α). It then follows from Theorem 2.2 that $G(\nu,\theta) \in X_N$ provided $(\nu,\theta) \in (0,\infty)\times X_N$. Finally since the exponential and sine functions in the definition of G are analytic it follows that G is analytic as an operator from $(0,\infty)\times X_1$ to X_1.

q.e.d.

In this thesis we shall concentrate on the local theory of small amplitude waves and in this case the spaces X_N which consist of C^1 functions are the most suitable. However there are a few global results concerning finite amplitude waves and in these cases it is more appropriate to work in the space of continuous functions. The reason for this is that the norm in the Banach space of continuous functions has a clear physical interpretation, representing the maximal angle between the wave profile and the horizontal. However, as is shown in Theorem 3.6 it makes no difference whether we seek small amplitude C^1 solutions or continuous solutions whose norm is less than $\pi/2$.

For $N \geq 1$ define

$$C_N^o(\mathbb{R}) = \{u \in C^o(\mathbb{R}) : u(t) = -u(-t) = u(t + \frac{2\pi}{N}) \; ; \; t \in \mathbb{R}\} \qquad (3.10)$$

and for d in the interval $(0, \pi/2)$ let $\mathcal{B}_N(d)$ denote the open ball in $C_N^o(\mathbb{R})$, centre the origin, radius $\pi/2-d$.

Before proving any of our main theorems we need the following results on conjugate functions. The first is due to Riesz and appears on page 253 of Zygmund, Volume I (1959).

THEOREM 3.3. _If_ $u \in C_1^0(\mathbb{R})$ _then_ $Cu \in L_p[-\pi,\pi]$ _for all_ $p \geq 1$ _and_

$$\int_{-\pi}^{\pi} |Cu(x)|^p \, dx \leq A_p \int_{-\pi}^{\pi} |u(x)|^p \, dx \tag{3.11}$$

where A_p _depends on_ p _only_.

The next result is a consequence of Theorem 2.11, page 254 of Zygmund, Volume I (1959).

THEOREM 3.4. _If_ $u \in \mathcal{B}_1(d)$ _and_ p _is such that_ $0 < p \leq \frac{\pi-d}{\pi-2d}$ (> 1), _then_

$$\int_{-\pi}^{\pi} \exp p|Cu(s)| \, ds \leq \frac{4\pi}{\sin d/2} \quad . \tag{3.12}$$

We can now prove a technical lemma.

LEMMA 3.5. _Let_ $\theta \in \mathcal{B}_1(d)$ _and let_ $-\tau$ _denote the conjugate of_ θ. _Then the functions_

$$\int_0^s \exp \tau(w) \, dw, \qquad \int_0^s \exp(-\tau(w)) \, dw \qquad \text{and}$$

$$\int_0^s \int_0^x \exp(\tau(x) + \tau(w)) \sin \theta(w) \, dw \, dx$$

are all Hölder continuous on $[-\pi,\pi]$ _with exponent_ α _where_ $\alpha = \frac{d}{\pi-d}$. _Further they are each bounded by a constant which depends only on_ d.

Proof. Define p as $\frac{\pi-d}{\pi-2d}$ so that $\alpha = \frac{d}{\pi-d} = \frac{p-1}{p}$. Then Theorem 3.4

states that

$$\int_{-\pi}^{\pi} \exp p\tau(w)\ dw \le \frac{4\pi}{\sin d/2}\ .$$

Now consider $\int_x^y \exp \tau(t)\ dt$ for $x, y \in [-\pi,\pi]$. By Hölder's inequality

$$\int_x^y \exp \tau(w)\ dw \le \left(\int_x^y dw \right)^{(p-1)/p} \left(\int_x^y \exp p\tau(w)\ dw \right)^{1/p}$$

$$\le |y-x|^{\alpha} \left(\frac{4\pi}{\sin d/2} \right)^{1/p} \qquad \text{since } \alpha = \frac{p-1}{p}\ . \tag{3.13}$$

This proves the results for $\int_x^y \exp \tau(w)\ dw$. A similar argument yields the same estimate for $\int_x^y \exp (-\tau(w))\ dw$, and an argument involving the estimate

$$\left| \int_0^x \exp \tau(w) \sin \theta(w)\ dw \right| < \int_0^x \exp \tau(w)\ dw < \frac{2^{1/p}\pi}{(\sin d/2)^{1/p}} \quad \text{yields}$$

that for $\int_0^s \int_0^x \exp(\tau(x) + \tau(w)) \sin \theta(w)\ dw\ dx$.

q.e.d

We now prove our first main result which shows that, under certain conditions we can seek either continuous or continuously differentiable solutions θ of $\gamma\theta = G(\nu, \theta)$.

THEOREM 3.6. Suppose that $\theta \in \mathcal{B}_1(d)$ and that for some positive numbers ν and γ, θ satisfies $\gamma\theta = G(\nu, \theta)$. Then $\theta \in C^{\infty}[-\pi,\pi]$.

Proof. Let $\alpha = \frac{d}{\pi-d}$, as in Lemma 3.5. Then if we write $G(\nu, \theta)$ as

$$\frac{\nu\left(\int_0^s \exp\tau(w)\,dw\int_0^\pi \exp(-\tau(w))\,dw - \int_0^s \exp(-\tau(w))dw\int_0^\pi \exp\tau(w)\,dw\right)}{\int_0^\pi \exp\tau(w)\,dw}$$

$$+ \int_0^s\int_0^x \exp(\tau(x) + \tau(w))\sin\theta(w)\,dw\,dx$$

$$- \frac{\int_0^s \exp\tau(w)\,dw\int_0^\pi\int_0^y \exp(\tau(y) + \tau(w))\sin\theta(w)\,dw\,dy}{\int_0^\pi \exp\tau(w)}\,, \tag{3.14}$$

it is clear from Lemma 3.5 that $\theta \in C^{0,\alpha}[-\pi,\pi]$. Theorem 3.1 then ensures that $\tau \in C^{0,\alpha}[-\pi,\pi]$ and hence

$$\int_0^s \exp(\pm\tau(w))\,dw \quad\text{and}\quad \int_0^s\int_0^x \exp(\tau(x) + \tau(w))\sin\theta(w)\,dw\,dx$$

are all $C^{1,\alpha}[-\pi,\pi]$. A "bootstrap" argument then yields the result.

q.e.d.

We can now prove some further results about the operator G when it acts on $(0,\infty)\times\mathcal{B}_1(d)$. First we need a preliminary lemma.

LEMMA 3.7. <u>Suppose</u> $\theta \in \mathcal{B}_1(d)$ <u>and let</u> $-\tau$ <u>denote the conjugate of</u> θ. <u>Then the functions</u>

$$\theta \to \begin{cases} \int_0^s \exp\tau(w)\,dw & (3.15)\\[2ex] \int_0^s \exp(-\tau(w))\,dw & (3.16)\\[2ex] \int_0^s\int_0^x \exp(\tau(x) + \tau(w))\sin\theta(w)\,dw\,dx & (3.17)\end{cases}$$

<u>are all real analytic from</u> $\mathcal{B}_1(d)$ <u>to</u> $C^o[-\pi,\pi]$.

<u>Proof</u>. It follows from Theorem 3.3 that $\tau(x) \in L^p[-\pi,\pi]$ for any $p \geq 1$. Then for any $n \in \mathbb{N}$

$$\int_0^s \exp\,\tau(w)\,dw = s + \int_0^s \tau(w)\,dw + \frac{1}{2!}\int_0^s \tau(w)^2\,dw + \ldots\ldots$$

$$\ldots + \frac{1}{(n-1)!}\int_0^s \tau(w)^{n-1}\,dw + \frac{1}{n!}\int_0^s\int_0^1 (1-t)^n\, e^{t\tau(w)}\,\tau(w)^n\,dt\,dw.$$

Then by Theorems 3.3 and 3.4

$$\int_0^s\int_0^1 (1-t)^n\, e^{t\tau(w)}\,\tau(w)^n\,dt\,dw = O(||\theta||^n) \quad \text{as } ||\theta|| \to 0.$$

This proves the lemma for (3.15), the results for the others follow similarly.

q.e.d.

We can now prove our next main theorem which is

THEOREM 3.8. <u>For any</u> $N \geq 1$, <u>if</u> $(\nu,\theta) \in (0,\infty)\times\mathcal{B}_N(d)$ <u>then</u> $G(\nu,\theta) \in C^o_N(\mathbb{R})$. <u>Moreover</u> $G : (0,\infty)\times\mathcal{B}_1(d) \to C^o_1(\mathbb{R})$ <u>is real analytic</u>.

<u>Proof</u>. It is clear by the same reasoning as in Theorem 2.2 that if $(\nu,\theta) \in (0,\infty)\times\mathcal{B}_N(d)$ then $G(\nu,\theta) \in C^o_N(\mathbb{R})$. Then it follows from Lemma 3.7 that $G : (0,\infty)\times\mathcal{B}_1(d) \to C^o_1(\mathbb{R})$ is real analytic.

q.e.d.

The last major result of this chapter concerns the compactness of the operator G. First we need to recall some definitions.

DEFINITION. Let F be a mapping from X to Y where Y is a Banach space and X a subset of a Banach space. Then F is said to be compact if and only if for each bounded sequence $\{x_n\}$ in X the sequence $\{F(x_n)\}$ contains a subsequence convergent in Y.

DEFINITION. Let $\mathcal{F}$ be a family of continuous functions defined on the interval $[a,b]$. Then $\mathcal{F}$ is said to be equicontinuous if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $f \in \mathcal{F}$ and $x, y \in [a,b]$ are such that $|x-y| < \delta$.

The next theorem is the Ascoli-Arzela theorem. A proof can be found on page 266 of Dunford and Schwarz, Part I (1958).

THEOREM 3.9. If $\mathcal{F}$ is a bounded, equicontinuous family in $C^0[a,b]$ then every sequence of functions in $\mathcal{F}$ contains a uniformly convergent subsequence.

We can now prove the following theorem, which will be needed when we apply Theorem 4.2 to discuss the global properties of solutions.

THEOREM 3.10. G is a compact operator from $(0,\infty)\times\mathcal{B}_1(d)$ to $C_1^0(\mathbb{R})$.

Proof. First apply the Cauchy-Schwarz inequality to $\exp \dfrac{\tau(w)}{2}$ and $\exp \dfrac{-\tau(w)}{2}$. This yields

$$\pi = \int_0^\pi dw \le \left(\int_0^\pi \exp \tau(w)\, dw\right)^{1/2}\left(\int_0^\pi \exp(-\tau(w))\, dw\right)^{1/2}$$

$$\le \frac{2^{1/2p}\pi^{\frac{1}{2}}}{(\sin d/2)^{1/2p}}\left(\int_0^\pi \exp \tau(w)\, dw\right)^{1/2} \qquad \text{where } p = \frac{\pi-d}{\pi-2d}$$

by Lemma 3.5.

So

$$\int_0^\pi \exp \tau(w)\ dw \geq \pi \frac{(\sin d/2)^{1/p}}{2^{1/p}} \quad . \tag{3.18}$$

Suppose then that $\{(\nu_n, \theta_n)\} \subset (0,\infty)\times\mathcal{B}_1(d)$ is a bounded sequence. So there is a constant K such that $|\nu_n| < K\ \forall\ n$. We seek to show that $G(\nu_n, \theta_n)(s)$ form a bounded, equicontinuous family. If we write $G(\nu,\theta)$ as in (3.14) then it is clear from (3.18) and the boundedness results of Lemma 3.5 that the sequence $\{G(\nu_n, \theta_n)(s)\}$ is bounded by a constant depending only on K and d. Further it follows from the results on Hölder continuity contained in Lemma 3.5 that for any $s, t \in \mathbb{R}$ and all $n \in \mathbb{N}$

$$|G(\nu_n, \theta_n)(s) - G(\nu_n, \theta_n)(t)| \leq A|t-s|^\alpha$$

where $\alpha = \dfrac{\pi}{\pi-d}$ and A depends only on K and d.

q.e.d.

CHAPTER IV

BIFURCATION THEORY

4.1 Introduction

Recall equation (N) which is written in abbreviated form as

$$\gamma\theta = G(\nu, \theta) \tag{N}$$

where γ , $\nu \in \mathbb{R}^+$ and $\theta \in X_1$. Recall that

$$\gamma = T/g \qquad \text{and} \qquad \nu = c^2/g \tag{4.1}$$

where T is the surface tension, c the phase speed and g the force of gravity. However, since g is supposed throughout to be a constant, γ and ν will be referred to as the surface tension and the square of the phase speed respectively. For the moment the surface tension will be regarded as a fixed parameter. Later we will regard γ as another parameter which can be varied and study the effect of this on the solution set.

It is clear that there is a known set of solutions

$$\{(\nu, 0) : \nu \in \mathbb{R}^+\} \quad \text{of} \quad (N).$$

These will be referred to as the trivial solutions and correspond physically to uniform horizontal laminar flow with flat free surface. Our goal is to seek information about the set of non-trivial solutions. First we need some notation and definitions.

DEFINITION. <u>For fixed</u> γ <u>suppose there is a curve of solutions</u> $C = \{(\nu(t), \theta(t)) : t \in (-\delta,\delta)\}$ <u>of</u> (N) <u>such that</u> $\theta(t) = 0$ <u>if and only if</u> $t = t_0 \in (-\delta,\delta)$. <u>Then the point</u> $(\nu(t_0),0)$ <u>is said to be a primary bifurcation point of</u> (N). C <u>is said to be a primary bifurcation curve or branch.</u>

(The concept of secondary bifurcation will be recalled in §5.1. When there is no risk of confusion, a primary bifurcation point will be referred to simply as a bifurcation point.)

It is a simple consequence of the Implicit Function Theorem that a necessary condition for $(\nu_0,0)$ to be a bifurcation point is that there should exist a $\theta_0 \neq 0$ such that $\gamma\theta_0 = G_\theta(\nu_0, 0)\theta_0$. (Here $G_\theta(\nu_0,0)$ denotes the Fréchet derivative of $G(\nu_0, \theta)$ with respect to θ evaluated at $\theta = 0$.) Values of ν with this property will be termed <u>eigenvalues</u> of the linearised operator and the corresponding θ will be termed <u>eigenfunctions</u>.

This condition, though necessary, is far from sufficient as the following example, to be found on page 193 of Krasnosel'skii (1964), shows.

Consider the equation $z = A(\mu, z)$ where $A : \mathbb{R}\times\mathbb{C} \to \mathbb{C}$ is defined by

$$A(\mu, z) = \mu z + i\mu z|z|^2. \tag{4.1}$$

The linearisation of A around $z = 0$ is

$$B(\mu, z) = \mu z.$$

Clearly $\mu = 1$ is an eigenvalue of the linearised operator. However whatever value μ may take, there are no non-zero solutions z of $z = A(\mu,z)$. For if $z \neq 0$ then $z = \mu z + i\mu z|z|^2$ implies

$1 = \mu + i\mu|z|^2$ which is impossible.

Nevertheless there are circumstances in which bifurcation from an eigenvalue can be guaranteed and such circumstances will be discussed in §4.3.

4.2 Eigenvalues of the Linearised Operator

The first step in the bifurcation analysis is to determine the eigenvalues and associated eigenfunctions of $\gamma\theta = G_\theta(\nu,0)\theta$. Since by Theorem 3.2 the operator $G(\nu,\cdot)$ is real analytic from X_1 to X_1 it is an elementary calculation to show that

$$G(\nu,\theta) = \nu A\theta + B\theta + H_1(\theta) + \nu H_2(\theta) \tag{4.3}$$

where

$$A\theta = \int_0^s \tau(w)\,dw\ , \tag{4.4}$$

$$B\theta = \frac{1}{\pi}\int_0^s\int_0^\pi\int_y^x \theta(w)\,dw\,dy\,dx \tag{4.5}$$

and

$$H_i(\theta) = O(||\theta||^2) \qquad \text{as } ||\theta|| \to 0 \qquad i = 1,2. \tag{4.6}$$

Hence the linearisation of (3.1) around $\theta = 0$ is

$$\gamma\theta = \nu A\theta + B\theta\ . \tag{L}$$

By expanding θ and τ in Fourier series it is straightforward to show that the eigenvalues ν_n of the linearised operator comprise the set $\{\gamma n + n^{-1} : n \in \mathbb{N}\}$ and the eigenfunction corresponding to $\nu_n = \gamma n + n^{-1}$ is $\sin ns$. Then, taking γ to be constant, the spectrum of the linearised operator is

$$\sigma_\gamma = \{\nu_k^\gamma = \gamma k + k^{-1} : k \in \mathbb{N}\}\ .$$

If the null space of the linearised operator corresponding to ν_k^γ is one dimensional then ν_k^γ will be termed a <u>simple eigenvalue</u>; if the null space is two dimensional ν_k^γ will be termed a <u>double eigenvalue</u>. It is then straightforward to show that if γ^{-1} is not the product of two distinct integers then every eigenvalue is simple, the eigenspace corresponding to ν_k^γ being spanned by sin ks. If however $\gamma^{-1} = MN$ for some $M, N \in \mathbb{N}$ $(M \neq N)$, then $\nu_M^\gamma = \nu_N^\gamma$ and there is at least one double eigenvalue in the spectrum, the corresponding null space being spanned by sin Ns and sin Ms. (There may be more than one double eigenvalue but there can only be a finite number of them. To see this suppose $\nu_k^\gamma = \nu_\ell^\gamma$, then

$$\frac{k}{MN} + k^{-1} = \frac{\ell}{MN} + \ell^{-1}.$$

Thus $MN = k\ell$ and so $\nu_k^\gamma = \nu_\ell^\gamma$ for those k and ℓ for which $k\ell = MN$ and there are only a finite number of such pairs.)

So whatever the value of γ, there are an infinite number of simple eigenvalues in the spectrum. It is the simple eigenvalue case which we shall now study.

4.3 Bifurcation from a simple eigenvalue

The theory surrounding bifurcation from a simple eigenvalue has been studied extensively and a description of the possible behaviour, both local and global, of the solution set which bifurcates from a simple eigenvalue may be obtained without difficulty.

Suppose that $N \geq 1$ and that ν_N is a simple eigenvalue of the linearised operator with corresponding eigenfunction sin Ns. Then we shall first determine the solution set of

$$\gamma\theta = G(\nu,\theta) \tag{N}$$

in a neighbourhood of $(\nu,\theta) = (\nu_N,0)$ where $\gamma \in \mathbb{R}^+$ is fixed, $\nu \in \mathbb{R}$ and $\theta \in X_N$. (The reason for choosing X_N rather than X_1 will be given shortly. Recall that by Theorem 3.2, $G(\cdot,X_N) \subset X_N$ and of course $\sin Ns \in X_N$.) Now with the notation of §4.2 let $\mathcal{N}(\nu_N A+B-\gamma I)$ and $\mathcal{R}(\nu_N A+B-\gamma I)$ denote the null space and range respectively of the linear map $\nu_N A+B-\gamma I : X_N \to X_N$. Then the following facts are true:

(1) $H_i(0) = H_i'(0) = 0 \quad (i = 1,2)$.

(2) $\mathcal{N}(\nu_N A+B-\gamma I) = \mathrm{sp}\{\sin Ns\}$.

(3) $X_N \backslash \mathcal{R}(\nu_N A+B-\gamma I)$ is one dimensional.

(4) $A(\sin Ns) \notin \mathcal{R}(\nu_N A+B-\gamma I)$.

(1 and 2 are obvious, 3 is true because

$$\mathcal{R}(\nu_N A+B-\gamma I) = \{u \in X_N : \int_0^{\pi/N} u(s)\sin Ns\, ds = 0\}$$

and 4 is true because

$$A(\sin Ns) = \frac{\sin Ns}{N} \notin R(\nu_N A+B-\gamma I).)$$

The fact that 1 - 4 hold means that the hypotheses of Theorem 2.4, page 330 of Crandall & Rabinowitz (1970) are satisfied and so we have the following result.

THEOREM 4.1. $(\nu_N,0)$ _is a bifurcation point for_ $\gamma\theta = G(\nu,\theta)$. _Moreover there is a_ $\delta > 0$ _and continuous functions_ $(\nu_N,\psi) : (-\delta,\delta)\times(-\delta,\delta) \to \mathbb{R}\times R(\nu_N A+B-\gamma I)$ _such that_ $\nu_N(0) = \nu_N$, $\psi(0) = 0$ _and there is a neighbourhood_ U _of_ $(\nu_N,0) \in \mathbb{R}\times X_N$ _such that any solution of_ (N) _in_ U _is either of the form_ $\{\nu_N(a), a \sin Ns + a\psi(a) : |a| < \delta\}$ _or is trivial_.

Remarks

1. If we seek solutions of (N) in a neighbourhood of $(\nu_N,0) \in \mathbb{R}\times X_1$ (rather than $\mathbb{R}\times X_N$), the facts 1 - 4 and Theorem 4.1 (with X_N replaced by X_1) are still true. The uniqueness result then means that all non-trivial solutions of (N) in a sufficiently small neighbourhood of $(\nu_N,0)$ in $\mathbb{R}\times X_1$ have been found and they all belong to $\mathbb{R}\times X_N$.
2. In Chapter V we shall see how to determine approximately these solution curves.

It will now be shown that the bifurcation curves of Theorem 4.1 are part of a global set of solutions. This follows essentially from the following theorem due to Rabinowitz (1970).

THEOREM 4.2. <u>Suppose</u> Ω <u>is an open bounded subset of</u> $\mathbb{R}\times E$ <u>where</u> E <u>is a Banach space and</u> $F(\lambda,u) : \overline{\Omega}\to E$ <u>is a continuous and compact operator</u>. <u>Suppose further</u> $F(\lambda,u) = \lambda Lu+H(\lambda,u)$ <u>where</u> L <u>is a linear operator and</u> $H(\lambda,u) = O(||u||^2)$ <u>uniformly for</u> λ <u>in bounded intervals as</u> $||u|| \to 0$. <u>Then if</u> λ_o <u>is such that the dimension of</u> $N(u-\lambda_o Lu)$ <u>is odd, there is a closed connected set</u> $\mathcal{C} \subset \overline{\Omega}$ <u>which consists of solution</u> (λ,u) <u>of</u> $u = F(\lambda,u)$ <u>such that</u> $(\lambda_o,0) \in \mathcal{C}$ <u>and</u> $\mathcal{C}$ <u>either</u>

(i) <u>meets</u> $\partial\Omega$ <u>or</u>

(ii) <u>meets</u> $(\hat{\lambda}, 0)$ <u>where</u> $\hat{\lambda} \neq \lambda_o$, $\hat{\lambda}$ <u>is an eigenvalue of</u> L <u>and</u> $(\hat{\lambda}, 0) \in \overline{\Omega}$.

We wish to apply this theorem to the equation $\gamma\theta = G(\nu,\theta)$. However, G does not satisfy the hypotheses of the theorem because

$$G(\nu,\theta) = \nu A\theta + B\theta + H(\nu,\theta).$$

Toland (1983) proved that provided A and B commute and $N(\gamma I-B)\cap N(A)=\{0\}$ the theorem is still true. We shall check these hypotheses and then show how Theorem 4.2 can be applied. First we must choose appropriate function spaces. Suppose $\nu_N = \gamma N + N^{-1}$ is a simple eigenvalue with corresponding eigenfunction sin Ns. Set $E = C_N^o(\mathbb{R})$ and $\Omega = (0,K)\times\mathcal{B}_N(d)$ where K is an arbitrary positive number. Then by Theorems 3.8 and 3.10 $G : \overline{\Omega} \to E$ is continuous and compact. Finally we must verify the hypotheses on G. First note that if $\theta \in \mathcal{B}_N(d)$ and $A\theta = 0$ then $\theta \equiv 0$. Hence $N(\gamma I-B) \cap N(A) = \{0\}$. To show that A and B commute suppose that $\theta \in \mathcal{B}_N(d)$ and $-\tau$ is its conjugate. Then $\theta \sim \sum_{n=1}^{\infty} a_n \sin ns$ and $\tau \sim \sum_{n=1}^{\infty} a_n \cos ns$ where $\{a_n\}$ is the sequence of Fourier coefficients of θ. Then

$$A\theta = \Sigma \frac{a_n}{n} \sin ns$$

and $$BA\theta = -\Sigma \frac{a_n}{n^3} \sin ns \quad ,$$

while $$B\theta = -\Sigma \frac{a_n}{n^2} \sin ns$$

$$AB\theta = -\Sigma \frac{a_n}{n^3} \sin ns.$$

Hence $AB\theta = BA\theta$ for all $\theta \in \mathcal{B}_N(d)$.

We now have the following corollary of Theorem 4.2.

COROLLARY 4.3. <u>There is a closed connected set</u> $C_{\nu_N} \subset (0,\infty)\times\mathcal{B}_N(d)$ <u>of solutions</u> (ν,θ) <u>of</u> $\gamma\theta = G(\nu,\theta)$ <u>such that</u> $(\nu_N,0) \in C_{\nu_N}$ <u>and</u> C_{ν_N} <u>either</u>

(i) <u>meets</u> $([0,\infty)\times\partial\mathcal{B}_N(d)) \cup (\{0\}\times\mathcal{B}_N(d))$ <u>or</u>

(ii) <u>for all</u> $\nu > \nu_N$ <u>there is a</u> $\theta \in \mathcal{B}_N(d)$ <u>such that</u> $(\nu,\theta) \in C_{\nu_N}$ <u>or</u>

(iii) C_{ν_N} <u>meets</u> $(\hat{\nu}, 0)$ <u>where</u> $\hat{\nu}(\neq \nu_N)$ <u>is an eigenvalue of the linearised equation</u>.

<u>Remark</u>. It is possible to apply Theorem 4.2 with $E = C_1^0(\mathbb{R})$ rather than $C_N^0(\mathbb{R})$ and obtain a result analogous to Corollary 4.3 giving the existence of a solution set $C^*_{\nu_N} \subset (0,\infty)\times\mathcal{B}_1(d)$ with similar properties to C_{ν_N}. By the uniqueness result of Theorem 4.1 there is a neighbourhood $\mathcal{O} \subset \mathbb{R}\times\mathcal{B}_1(d)$ of $(\nu_N,0)$ such that $\mathcal{O} \cap C_{\nu_N} = \mathcal{O} \cap C^*_{\nu_N}$ and clearly $C_{\nu_N} \subseteq C^*_{\nu_N}$. Since there is no uniqueness result contained in Corollary 4.3 it is possible that $C^*_{\nu_N} \setminus C_{\nu_N} \neq \emptyset$. However obviously if it is non-empty, $C^*_{\nu_N} \setminus C_{\nu_N}$ must either itself satisfy one of the alternatives of Corollary 4.3 or be connected to $C_{\nu_N} \setminus \mathcal{O} \cap C_{\nu_N}$.

4.4 Bifurcation from a double eigenvalue

We now consider bifurcation from a double eigenvalue: i.e. the situation in which $\gamma \in \Gamma$ where

$$\Gamma = \{\frac{1}{MN} : M, N \in \mathbb{N} , M \neq N \} .$$

The spectrum σ_γ of the linearised problem is (for fixed γ)

$$\sigma_\gamma = \{\nu_k^\gamma = \gamma k + k^{-1} : k \in \mathbb{N}\},$$

and in this case σ_γ contains at least one double eigenvalue. There is no general theory concerning bifurcation from a double eigenvalue and indeed the example of Krasnosel'skii in which the eigenspace corresponding to $\mu = 1$ has dimension two shows that bifurcation need not occur. This problem may be approached by the method of Lyapunov-Schmidt and we shall show that in this case a multiplicity of solutions bifurcates from each double eigenvalue. (Precisely how many depends on the values of M and N.) We shall also consider what happens when γ is perturbed from one of the values in Γ. In this situation the corresponding double eigenvalue splits into two simple eigenvalues and we know from the previous section that from each of these there bifurcates a curve of solutions which is unique in a neighbourhood of the bifurcation point. However we shall show that secondary bifurcation may occur on these branches and also that a double eigenvalue may be regarded as the limiting case of two simple eigenvalues.

First we need some notation: let $M, N \in \mathbb{N}$ be fixed with $M > N$ and set

$$\gamma_{NM} = \frac{1}{MN} , \qquad \nu_{NM} = \frac{1}{N} + \frac{1}{M} .$$

Now we introduce the variables α and β by putting

$$\nu = \nu_{NM}+\alpha \quad , \qquad \gamma = \gamma_{NM}+\beta \quad .$$

The equation (N) takes the form

$$(\gamma_{NM}+\beta)\theta = G(\nu_{NM}+\alpha, \theta)$$

and we seek solutions (α,β,θ) in a neighbourhood of the origin in $\mathbb{R}\times\mathbb{R}\times X_1$.

Remark. Of particular physical interest is the bifurcation which may occur from the smallest eigenvalue, since this corresponds to the slowest phase speed at which bifurcation may take place. A simple calculation shows that for arbitrary $\gamma > 0$ the smallest element of σ_γ is

$$\begin{aligned} \gamma k + k^{-1} \quad & \text{if} \quad \gamma k(k+1) > 1 \quad \text{and} \\ \gamma(k+1)+(k+1)^{-1} \quad & \text{if} \quad \gamma k(k+1) < 1 , \end{aligned}$$

where $\gamma = [k^{-\frac{1}{2}}]$ is the greatest integer not exceeding $k^{-\frac{1}{2}}$. In both cases it is simple. If however $\gamma k(k+1) = 1$, i.e. $\gamma = k^{-1}(k+1)^{-1}$ then the smallest member of σ_γ is

$$\nu_k = \frac{1}{k} + \frac{1}{k+1}$$

and it is a double eigenvalue, with corresponding eigenspace spanned by sin ks and sin(k+1)s. In other words, when M = N+1 the smallest element of the spectrum is a double eigenvalue. This will be referred to again in §6.2(c).

4.5 The Method of Lyapunov-Schmidt

The Lyapunov-Schmidt procedure provides the key to the study of (N). It is essentially an application of the Implicit Function Theorem by which the infinite dimensional problem (N) may be reduced to one of finite dimensions. We now state this theorem since it is used a number of times in this thesis, both here and in Chapter V. For a proof see Dieudonné (1960) page 265.

THE IMPLICIT FUNCTION THEOREM. Let A, B, C be Banach spaces and F be a continuously differentiable mapping from $A \times B$ into C. Let $(x_o, y_o) \in A \times B$ be such that $F(x_o, y_o) = 0$ and that $F_y(x_o, y_o)$ is a linear homeomorphism of B onto C. Then there is a neighbourhood U of x_o and a continuous function y from U to B such that $y(x_o) = y_o$, $F(x_o, y(x_o)) = 0$ and if $F(x,y) = 0$ and $x \in U$ then $y = y(x)$.

Before describing the Lyapunov-Schmidt method in detail we need some definitions. Let

$$E_{NM} = \mathrm{sp}\{\sin Ns, \sin Ms\}.$$

Then E_{NM} is the solution space of the linear equation $\gamma_{NM}\theta = G_\theta(\nu_{NM}, 0)\theta$ where the right-hand side is regarded as an operator from X_1 to X_1. Now any element of E_{NM} is of the form $a \sin Ns + b \sin Ms$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and sometimes we shall identify E_{NM} with $\mathbb{R}^2$ via the identification of $a \sin Ns + b \sin Ms$ with (a,b). Let F_{NM} be the closure in X_1 of

$$\{\sin ks : k \in \mathbb{N} \setminus \{M, N\}\}.$$

Then $X_1 = E_{NM} \oplus F_{NM}$. Finally denote by P_{NM} and Q_{NM} the projections of X_1 onto E_{NM} and F_{NM} respectively (so $Q_{NM} = I - P_{NM}$ where $I : X_1 \to X_1$ is the identity operator). The Lyapunov-Schmidt procedure is then to replace (N) by the equivalent system

$$(\gamma_{NM}+\beta)P_{NM}\theta = P_{NM}G(\nu_{NM}+\alpha,\ P_{NM}\theta+Q_{NM}\theta), \tag{4.7a}$$

$$(\gamma_{NM}+\beta)Q_{NM}\theta = Q_{NM}G(\nu_{NM}+\alpha,\ P_{NM}\theta+Q_{NM}\theta), \tag{4.7b}$$

which, on writing $P_{NM}\theta = e$ and $Q_{NM}\theta = f$ becomes

$$(\gamma_{NM}+\beta)e = P_{NM}G(\nu_{NM}+\alpha,\ e+f), \tag{4.8a}$$

$$(\gamma_{NM}+\beta)f = Q_{NM}G(\nu_{NM}+\alpha,\ e+f). \tag{4.8b}$$

Now clearly

$$Q_{NM}G_\theta(\nu_{NM},0) : F_{NM} \to F_{NM}$$

is a linear homeomorphism and hence we can apply the Implicit Function Theorem to (4.8b) and deduce the existence of a neighbourhood Ω of the origin in $\mathbb{R}\times\mathbb{R}\times E_{NM}$ and a function $z(\alpha,\beta,e) : \Omega \to F_{NM}$ such that $f = z(\alpha,\beta,e)$ is the unique solution of (4.8b) for $(\alpha,\beta,e) \in \Omega$. Then substituting this into (4.8a) we obtain

$$\beta e = P_{NM}G(\nu_{NM}+\alpha,\ e+z(\alpha,\beta,e)) - \gamma_{NM}e . \tag{$\mathcal{B}$}$$

This equation is called the <u>bifurcation equation</u>. Then, putting $e = a \sin Ns + b \sin Ms$, it can be written as a pair of algebraic equations

$$\beta a = f_1(\alpha, \beta, a, b), \tag{$\mathcal{B}_1$}$$

$$\beta b = f_2(\alpha, \beta, a, b), \tag{$\mathcal{B}_2$}$$

where $f_1(\alpha, \beta, a, b) = \langle G(\nu_{NM}+\alpha, e+z(\alpha, \beta, e)) - \gamma_{NM}e, \sin Ns\rangle$,

and $f_2(\alpha, \beta, a, b) = \langle G(\nu_{NM}+\alpha, e+z(\alpha, \beta, e)) - \gamma_{NM}e, \sin Ms\rangle$.

Here $\langle,\rangle$ denotes the usual L_2 inner product, and $(\alpha, \beta, a, b) \in U$ which is a neighbourhood of the origin in $\mathbb{R}^4$ (identifying E_{NM} with $\mathbb{R}^2$).

Hence the bifurcation equation is a system of two scalar equations and is equivalent to the infinite dimensional problem, in that to any solution (α, β, a, b) of $(\mathcal{B})$, there corresponds a solution of $(\mathcal{N})$, namely

$$(\alpha, \beta, \theta) = (\alpha, \beta, a \sin Ns + b \sin Ms + z(\alpha, \beta, a, b)).$$

(We shall write $z(\alpha,\beta,e)$ as $z(\alpha,\beta,a,b)$ if $e = a \sin Ns + b \sin Ms$.)

4.6 Properties of the Bifurcation Equations

In this section we will draw on the symmetry and invariance results observed in §2.4 to deduce various properties enjoyed by the bifurcation equation $(\mathcal{B})$. Notice first that since by Theorem 3.2, $G(\cdot, X_M) \subset X_M$, the question of bifurcation from $\alpha = M\beta$ may be studied in the context of X_M as well as in that of X_1. In this context, the solution space of the linearised equation $\gamma_{NM}\theta = G_\theta(\nu_{NM}, 0)\theta$ is one dimensional, spanned by sin Ms. Define $\tilde{Z}_M$ to be the closed subspace of X_M such that $X_M = \mathrm{sp}\{\sin Ms\} \oplus \tilde{Z}_M$ and let $\tilde{P}_M$ be the projection of X_M onto sp{sin Ms}.

Then any solution $\theta \in X_M$ of (N) may be written as $b \sin Ms + \tilde{f}$ where $\tilde{f} \in \tilde{Z}_M$. Thus (N) is equivalent to the two equations

$$(\gamma_{NM}+\beta)b \sin Ms = \tilde{P}_M G(\nu_{NM}+\alpha,\ b \sin Ms + \tilde{f}) \tag{4.9a}$$

$$(\gamma_{NM}+\beta)\tilde{f} = (I - \tilde{P}_M)G(\nu_{NM}+\alpha,\ b \sin Ms + \tilde{f}). \tag{4.9b}$$

By the Implicit Function Theorem there is a neighbourhood $\tilde{\Omega}$ of the origin in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and a function $\tilde{z}(\alpha, \beta, b) : \tilde{\Omega} \to \tilde{Z}_M$ such that $\tilde{f} = \tilde{z}(\alpha,\beta,b)$ is the unique solution of (4.9b) for $(\alpha,\beta,b) \in \tilde{\Omega}$. Then by Theorem 2.3 it follows that

$$G(\nu_{NM}+\alpha,\ b \sin Ms + \tilde{z}(\alpha, \beta, b)) \in X_M, \quad (\alpha, \beta, b) \in \tilde{\Omega}$$

and hence

$$\langle G(\nu_{NM}+\alpha,\ b \sin Ms + \tilde{z}(\alpha, \beta, b)), \sin Ns\rangle = 0,$$

since if $\phi \in X_M$, then $<\phi, \sin Ns> = 0$, (recall $M > N$). Hence by (4.9b)

$$(\gamma_{NM}+\beta)\tilde{z}(\alpha,\beta,b) = (I-P_{NM})G(\nu_{NM}+\alpha,\ b \sin Ms + \tilde{z}(\alpha,\beta,b))$$

for all $(\alpha, \beta, b) \in \tilde{\Omega}$.

It is thus a consequence of uniqueness that

$$\tilde{z}(\alpha, \beta, b) = z(\alpha, \beta, 0, b).$$

In other words

$$z(\alpha, \beta, 0, b) \in X_M$$

and indeed

$$<z(\alpha, \beta, 0, b), \sin Ms> = 0.$$

Then by its definition we have

$f_1(\alpha, \beta, 0, b) = 0$ for all $(\alpha, \beta, 0, b) \in U$. A similar argument yields that

$$f_2(\alpha, \beta, a, 0) = 0 \quad \text{for all } (\alpha,\beta, a, 0) \in U$$

provided that $M \neq kN$ for any $k \in \mathbb{N}$.

However, if $M = kN$ for some $k \in \mathbb{N}$ then the argument used to prove $f_2(\alpha, \beta, a, 0) = 0$ breaks down. For although it is true in this case that $z(\alpha, \beta, a, 0) \in X_N$, we also have that $\sin Ms \in X_M \subset X_N$ and thus $\sin Ms$ is not orthogonal (in the L_2 sense) to every element in X_N. Therefore one would expect the series expansion of

$$G(\nu_{NM}+\alpha,\ a \sin Ns + z(\alpha,\beta,a,0))$$

to contain a term in $a^M \sin Ms$. This has indeed shown to be true when $k = 2, 3, 4, 5$ (see §4.9) and we suspect it to be true in the general case although a proof is lacking.

Note also from the above reasoning that $b \sin Ms + z(\alpha,\beta,0,b) \in X_M$ and $a \sin Ns + z(\alpha,\beta,a,0) \in X_N$ for all $(\alpha,\beta,a,b) \in U$. Conversely if $a \sin Ns + b \sin Ms + z(\alpha,\beta,a,b) \in X_N$ and $M \neq kN$ for any $k \in \mathbb{N}$ then $b = 0$ while if $a \sin Ns + b \sin Ms + z(\alpha,\beta,a,b) \in X_M$ then $a = 0$. In particular, if $M \neq kN$ for all $k \in \mathbb{N}$, then solutions θ of (N) in X_M or X_N arise from solutions of (B) with $ab = 0$. This observation will be seen to be crucial in §5.3 when we come to discuss the existence of bifurcation points.

We now give some of the implications for the bifurcation equations and their solutions of the theorems in Chapters II and III concerning the symmetries and invariant subspaces of the operator G. Let K be any common factor M and N (K may be 1). Recall from Theorems 2.2 and 3.2 that $G(\cdot\,, X_K) \subset X_K$ and also $E_{NM} \subset X_K \subseteq X_1$; hence the equation (N) may be posed in $\mathbb{R}\times\mathbb{R}\times X_K$ rather than $\mathbb{R}\times\mathbb{R}\times X_1$ and precisely the same bifurcation equations are obtained in each case. Hence by the uniqueness results, all solutions $\theta = e+z(\alpha,\beta,e)$ which arise as a result of this analysis lie in X_K and correspond to capillary-gravity waves of greatest minimal period $2\pi/K$.

It was proved in Theorems 2.3 and 3.2 that if $\theta \in X_K$ then $S_K G(\nu_{NM}+\alpha, \theta) = G(\nu_{NM}+\alpha, S_K\theta)$. In other words S_K commutes with $G(\nu_{NM}+\alpha, \cdot)$ on X_K. Then since Q_{NM} obviously commutes with S_K on X_K it follows that

$$S_K Q_{NM} G(\nu_{NM}+\alpha, e+z) = Q_{NM} G(\nu_{NM}+\alpha, S_K(e+z))$$

whence, again by uniqueness,

$$z(\alpha, \beta, S_K e) = S_K z(\alpha, \beta, e).$$

Hence on writing e as a sin Ns + b sin Ms we obtain

$$z(\alpha, \beta, (-1)^{N/K}a, (-1)^{M/K}b) = S_K z(\alpha, \beta, a, b),$$

where K is any common divisor of M and N.
An analogous argument gives that

$$S_M Q_{NM} G(\nu_{NM}+\alpha, b \sin Ms + z(\alpha,\beta,0,b)) = Q_{NM} G(\nu_{NM}+\alpha, -b \sin Ms + S_M z(\alpha,\beta,0,b)),$$

whence $z(\alpha, \beta, 0, -b) = S_M z(\alpha, \beta, 0, b)$ and similarly $z(\alpha, \beta, -a, 0) = S_N z(\alpha, \beta, a, 0)$.

When these results are substituted into the bifurcation equations we obtain for any common divisor K of M and N

$$f_1(\alpha, \beta, (-1)^{N/K}a, (-1)^{M/K}b) = (-1)^{N/K} f_1(\alpha, \beta, a, b), \tag{4.10a}$$

$$f_2(\alpha, \beta, (-1)^{N/K}a, (-1)^{M/K}b) = (-1)^{M/K} f_2(\alpha, \beta, a, b), \tag{4.10b}$$

$$f_1(\alpha,\beta,\ -a,\ 0) = -f_1(\alpha,\beta,\ a,\ 0),$$

$$f_2(\alpha,\beta,\ 0,\ -b) = -f_2(\alpha,\beta,\ 0,\ b).$$

It is easy to see that no information is lost by taking K to be the highest common factor of M and N.

It should be noted at this stage that any additional solutions which arise as a result of these observations do not yield different solutions of the capillary-gravity wave problem. For instance if $\theta \in X_K$ is a solution of (N), then even though $S_K\theta$ may be a distinct solution, it follows from Theorem 2.4 that these are different parameterisations of the same capillary-gravity wave. The question of the number of distinct solutions to the capillary-gravity wave problem will be dealt with more fully in Chapter VI.

4.7 Implementation of the Method

The Lyapunov-Schmidt reduction will now be carried out. This task is of an elementary nature but involves formidable calculations because, as will soon become apparent, the bifurcation equations may be degenerate at the quadratic as well as at the linear level. The first step is to notice that since $G(\nu_{NM} + \alpha,\theta)$ is a real analytic operator from $\mathbb{R}\times X_1$ to X_1 it is a matter of calculation to show that it may be written as

$$(\nu_{NM}+\alpha)A\theta + B\theta + Q(\theta) + \nu_{NM}C_1(\theta) + C_2(\theta) + H(\alpha,\theta),\ \theta \in X_1 \tag{4.11}$$

where

$$A\theta(s) = \int_0^s \tau(x)\ dx \tag{4.12}$$

$$B\theta(s) = \frac{1}{\pi}\int_0^s\int_0^\pi\int_y^x \theta(w)\ dw\ dy\ dx \tag{4.13}$$

$$Q\theta(s) = \frac{1}{\pi}\int_0^s\int_0^\pi\int_y^x \theta(w)[\tau(w) + \tau(x) + \tau(y)]\ dw\ dy\ dx \tag{4.14}$$

$$C_1\theta(s) = \frac{1}{6\pi}\int_0^s\int_0^\pi (\tau(x)^3 - \tau(w)^3)\ dw\ dx, \tag{4.15}$$

$$\begin{aligned} C_2\theta(s) = &- \frac{1}{6\pi}\int_0^s\int_0^\pi\int_y^x \theta(w)^3\ dw\ dy\ dx \\ &- \frac{1}{2\pi^2}\left\{\int_0^\pi \tau(w)^2\ dw\right\}\left\{\int_0^s\int_0^\pi\int_y^x \theta(w)\ dw\ dy\ dx\right\} \\ &+ \frac{1}{2\pi}\int_0^s\int_0^\pi\int_y^x [\tau(x) + \tau(y) + \tau(w)]^2\theta(w)\ dw\ dy\ dx, \end{aligned} \tag{4.16}$$

and $H(\alpha,\theta) = O(||\theta||^3(||\theta|| + |\alpha|))$ as $||\theta||,\alpha \to 0$.

It is therefore clear that (4.8b) may be re-written as

$$((\gamma_{NM}+\beta)I - (\nu_{NM}+\alpha)A-B)f = Q_{NM}\{Q(e+f)+\nu_{NM}C_1(e+f)+C_2(e+f)+H(\alpha,e+f)\} \quad (4.17)$$

Recall that by the Implicit Function Theorem this equation has a unique solution $f = z(\alpha,\beta,e)$ for (α,β,e) in a neighbourhood Ω of $(0, 0, 0)$. Let $D^N z[\alpha,\beta,e]$ denote the N^{th} derivative of the function z at a point $(\alpha,\beta,e) \in \Omega$. Then putting $f = z(\alpha,\beta,e)$ in (4.17) and differentiating yields

$$(\gamma_{NM}I - \nu_{NM}A - B) \circ D^1 z[0, 0, 0](\alpha, \beta, e) = 0$$

for all $(\alpha, \beta, e) \in \Omega$

whence $D^1 z[0, 0, 0]$ is the zero operator from Ω into F_{NM}, since $(\gamma_{NM}I - \nu_{NM}A - B)$is injective on F_{NM}.

Now a second differentiation yields that

$$(\gamma_{NM}I-\nu_{NM}A-B) \circ D^2 z[0, 0, 0](\alpha, \beta, e)(\tilde{\alpha}, \tilde{\beta}, \tilde{e})$$

$$= 2Q_{NM}q(e,\tilde{e}), \ (\alpha, \beta, e), \ (\tilde{\alpha}, \tilde{\beta}, \tilde{e}) \in \Omega , \quad (4.18)$$

where for any $\theta_1, \theta_2 \in X_1$,

$$q(\theta_1, \theta_2) = \frac{1}{2\pi} \left\{ \int_0^s \int_0^\pi \int_y^x \theta_1(w)[\tau_2(x)+\tau_2(y)+\tau_2(w)] \, dw \, dy \, dx \right.$$

$$\left. + \int_0^s \int_0^\pi \int_y^x \theta_2(w)[\tau_1(x)+\tau_1(y)+\tau_1(w)] \, dw \, dy \, dx \right\}. \quad (4.19)$$

Note that, for any $\theta \in X_1$, $Q(\theta) = q(\theta,\theta)$. Hence the Taylor polynomial of z of order two about (0, 0, 0) $\in \mathbb{R}\times\mathbb{R}\times E_{NM}$ is

$$\{(\gamma_{NM}I - \nu_{NM}A-B)^{-1}\circ Q_{NM}\}Q(e), \qquad (\alpha, \beta, e) \in \Omega \tag{4.20}$$

where the inverse operator is considered as an operator on F_{NM}.

If the bifurcation equation $(\mathcal{B})$ is written out in full we obtain the following equation for $e \in E_{NM}$

$$\begin{aligned}\beta e = \alpha Ae + P_{NM}\{&Q(e+z(\alpha,\beta,e)) + \nu_{NM}C_1(e+z(\alpha,\beta,e)) + C_2(e+z(\alpha,\beta,e)) \\ &+ H(\alpha, e + z(\alpha,\beta,e))\} \quad ,\end{aligned} \tag{4.21}$$

using (4.20) this can be written as

$$\begin{aligned}\beta e = \alpha Ae + P_{NM}\{&Q(e) + 2q(e,(\gamma_{NM}I-\nu_{NM}A-B)^{-1}\circ Q_{NM}\circ Q(e)) \\ &+ \nu_{NM}C_1(e) + C_2(e)\} + \tilde{H}(\alpha, \beta, e)\end{aligned} \tag{4.22}$$

where $\tilde{H}$ maps a neighbourhood of the origin in $\mathbb{R}\times\mathbb{R}\times E_{NM}$ into E_{NM}, is $O(||e||^3(|\alpha|+|\beta|+||e||))$ at zero, and is real analytic.

Now we wish to interpret (4.22) as an equation for the components of e with respect to the basis {sin Ns, sin Ms} of E_{NM}. Let

$$e = a \sin Ns + b \sin Ms \quad . \tag{4.23}$$

Then it is clear that

$$\alpha Ae = \frac{a\alpha}{N}\sin Ns + \frac{b\alpha}{M}\sin Ms \ . \tag{4.24}$$

The higher order terms are a great deal more complicated and we shall consider them individually.

4.7(i) The Quadratic Term

In equation (4.22) the quadratic term is

$$P_{NM}Q(e) = P_{NM}Q(a \sin Ns + b \sin Ms), \tag{4.25}$$

but it is convenient at this stage to evaluate a slightly more general expression, namely

$$q(\sin Ns, \sin Ms). \tag{4.26}$$

The evaluation of (4.25) will then be immediate.

Now for all M, N $\in \mathbb{N}$,

$$\frac{1}{2\pi}\left\{\int_0^s\int_0^\pi\int_y^x \sin Nw(\cos Mx + \cos My + \cos Mw)\,dw\,dy\,dx\right\}$$

$$= \begin{cases} -\frac{1}{4}\left(\frac{2N+M}{N(N+M)^2}\sin(N+M)s + \frac{2N-M}{N(N-M)^2}\sin(N-M)s\right), & M \neq N \\ -\frac{3}{16N^2}\sin 2Ns, \quad M = N. & \end{cases}$$

Hence

$q(\sin Ns, \sin Ms)$

$$= \begin{cases} -\dfrac{1}{4}\left(\dfrac{N^2+4MN+M^2}{MN(M+N)^2}\sin(M+N)s + \dfrac{M+N}{MN(N-M)}\sin(N-M)s\right), & M \neq N \\ -\dfrac{3}{8N^2}\sin 2Ns \qquad M = N, \end{cases} \tag{4.27}$$

and so when $e = a \sin Ns + b \sin Ms$ we obtain

$$Q(e) = -\frac{3a^2}{8N^2}\sin 2Ns - \frac{3b^2}{8M^2}\sin 2Ms$$
$$- \frac{ab}{2}\left(\frac{N^2+4MN+M^2}{MN(N+M)^2}\sin(M+N)s + \frac{M+N}{MN(M-N)}\sin(M-N)s\right). \tag{4.28}$$

Therefore

$$P_{NM}Q(a \sin Ns + b \sin Ms) = 0 \in E_{NM}, \quad \text{if } M \neq 2N \tag{4.29}$$

and

$$P_{NM}Q(a \sin Ns + b \sin Ms) = -\frac{3a^2}{8N^2}\sin 2Ns - \frac{3ab}{4N^2}\sin Ns \text{ if } M = 2N. \tag{4.30}$$

(Recall that $M > N$.)
Note therefore that only when $\gamma = \frac{1}{2N^2}$ and $\nu = \frac{3}{2N}$ $N \in \mathbb{N}$ will the bifurcation equations contain quadratic terms. The bifurcation phenomenon is not typically quadratic even in this case, since the quadratic term $P_{NM}Q(e)$ is degenerate, vanishing along the ray $a = 0$ in E_{NM}.

4.7(ii) The Cubic Terms

(a) The term $2P_{NM}q(e,(\gamma_{NM}I-\nu_{NM}A-B)^{-1}\circ Q_{NM}\circ Q(e))$.

Before evaluating this it will be useful to record that if $k \notin \{N,M\}$ we have

$$(\gamma_{NM}I - \nu_{NM}A - B)\sin ks$$

$$= \left(\gamma_{NM} - \frac{\nu_{NM}}{k} + \frac{1}{k^2}\right)\sin ks$$

$$= \left\{\frac{1}{NM} - \left(\frac{1}{N} + \frac{1}{M}\right)\frac{1}{k} + \frac{1}{k^2}\right\} \sin ks$$

$$= \left\{\frac{(k-N)(k-M)}{NMk^2}\right\} \sin ks, \tag{4.31}$$

and so $$(\gamma_{NM}I - \nu_{NM}A - B)^{-1}\sin ks = \left(\frac{NMk^2}{(k-N)(k-M)}\right)\sin ks, \quad k \notin \{N,M\}. \tag{4.32}$$

Now from (4.28) it is clear that

$$Q_{NM}\circ Q(e) = \begin{cases} Q(e) & \text{if } M \neq 2N, \\ -\dfrac{3b^2}{32N^2}\sin 4Ns - \dfrac{13ab}{36N^2}\sin 3Ns & \text{if } M = 2N \ . \end{cases}$$

A calculation using (4.32) now yields that

$$(\gamma_{NM}I - \nu_{NM}A - B)^{-1} \circ Q_{NM} \circ Q(e)$$

$$= \begin{cases} -\dfrac{3Ma^2}{2(2N-M)}\sin 2Ns - \dfrac{3Nb^2}{2(2M-N)}\sin 2Ms \\[2ex] -\dfrac{(M^2+4MN+N^2)}{2MN}ab\sin(M+N)s + \dfrac{(M^2-N^2)}{2N(M-2N)}ab\sin(M-N)s, \quad M \neq 2N \\[2ex] -\dfrac{b^2}{2}\sin 4Ns - \dfrac{13}{4}ab\sin 3Ns, \qquad M = 2N \quad . \end{cases} \tag{4.33}$$

Therefore by (4.27) and (4.33) we obtain

$$2P_{NM}q(e, (\gamma_{NM}I - \nu_{NM}A - B)^{-1} \circ Q_{NM} \circ Q(e))$$

$$= \begin{cases} \left\{\dfrac{9Ma^3}{8N^2(2N-M)} + \dfrac{ab^2}{4}\left(\dfrac{(2M+N)(M^2+4MN+N^2)}{M^2N^2(M+N)} - \dfrac{(2M-N)(M+N)}{MN^2(M-2N)}\right)\right\}\sin Ns \\[2ex] + \left\{\dfrac{9Nb^3}{8M^2(2M-N)} + \dfrac{a^2b}{4}\left(\dfrac{(M+2N)(M^2+4MN+N^2)}{M^2N^2(M+N)} - \dfrac{(M^2+2MN-2N^2)(M+N)}{M^2N^2(M-2N)}\right)\right\}\sin Ms, \\[1ex] \qquad\qquad\qquad\qquad\qquad\qquad M \neq 2N, 3N \\[2ex] \dfrac{65ab^2}{48N^2}\sin Ns + \left(\dfrac{3b^3}{32N^2} + \dfrac{13a^2b}{12N^2}\right)\sin 2Ns, \qquad M = 2N \\[2ex] -\left(\dfrac{27a^3}{8N^2} + \dfrac{39a^2b}{8N^2} + \dfrac{43ab^2}{72N^2}\right)\sin Ns \\[2ex] + \left(\dfrac{b^3}{40N^2} - \dfrac{49a^2b}{72N^2} - \dfrac{13a^3}{8N^2}\right)\sin 3Ns, \qquad M = 3N. \end{cases} \tag{4.34}$$

(b) <u>The term</u> $\nu_{NM}P_{NM}C_1(e)$.

From (4.15) it follows that

$$\frac{d}{ds}(C_1(\theta))(s) = \frac{1}{6}\left\{\tau(s)^3 - \frac{1}{\pi}\int_0^{\pi} \tau(w)^3\, dw\right\} .$$

From the definition of conjugate it is clear that when

$$\theta(s) = a \sin Ns + b \sin Ms$$

$$\tau(s) = a \cos Ns + b \cos Ms,$$

so $\tau(s)^3 = a^3\cos^3 Ns + 3a^2 b\cos^2 Ns \cos Ms + 3ab^2 \cos Ns \cos^2 Ms + b^3\cos^3 Ms$

$$= \frac{a^3}{4}(\cos 3Ns + 3 \cos Ns)$$

$$+ \frac{3a^2b}{4}(2 \cos Ms + \cos(M+2N)s + \cos(M-2N)s)$$

$$+ \frac{3ab^2}{4}(2 \cos Ns + \cos(2M+N)s + \cos(2M-N)s)$$

$$+ \frac{b^3}{4}(\cos 3Ms + 3 \cos Ms) .$$

Whence it follows that

$$P_{NM}C_1(e) = \begin{cases} \dfrac{1}{8}\left\{\dfrac{a^3+2ab^2}{N} \sin Ns + \dfrac{b^3+2a^2b}{M} \sin Ms\right\}, & M \neq 3N, \\[2ex] \dfrac{1}{4N}\left(\dfrac{a^3}{2} + \dfrac{a^2b}{2} + ab^2\right) \sin Ns + \dfrac{1}{12N}\left(\dfrac{a^3}{6} + a^2b + \dfrac{b^3}{2}\right) \sin 3Ns, & M = 3N . \end{cases} \qquad (4.35)$$

(c) <u>The term</u> $P_{NM}C_2(e)$

From (4.16) we can calculate $P_{NM}C_2(e)$ when $e = a \sin Ns + b \sin Ms$ as follows:

The first term is

$$-\frac{1}{6\pi}\int_0^s\int_0^\pi\int_y^x (a \sin Nw + b \sin Mw)^3 \, dw \, dy \, dx \quad .$$

Now $(a \sin Nw + b \sin Mw)^3$

$$= a^3\sin^3 Nw + 3a^2 b\sin^2 Nw \sin Mw + 3ab^2 \sin Nw \sin^2 Mw + b^3\sin^3 Mw$$

$$= \frac{a^3}{4}(3 \sin Nw - \sin 3Nw) + \frac{3a^2b}{4}(2 \sin Mw - \sin(M+2N)w - \sin(M-2N)w)$$

$$+ \frac{3ab^2}{4}(2 \sin Nw - \sin(2M+N)w - \sin(N-2M)w)$$

$$+ \frac{b^3}{4}(3 \sin Mw - \sin 3Mw).$$

So it follows that

$$P_{NM}\left(-\frac{1}{6\pi}\int_0^s\int_0^\pi\int_y^x (a \sin Nw + b \sin Mw)^3 \, dw \, dy \, dx\right)$$

$$= \begin{cases} \dfrac{1}{8}\left\{\left(\dfrac{a^3+2ab^2}{N^2}\right)\sin Ns + \left(\dfrac{b^3+2a^2b}{M^2}\right)\sin Ms\right\}, & M \neq 3N \\[2ex] \dfrac{1}{8N^2}(a^3-a^2b+2ab^2)\sin Ns + \dfrac{1}{36N^2}\left(\dfrac{b^3}{2} - \dfrac{a^3}{6} + a^2b\right)\sin 3Ns & \\ & M = 3N. \end{cases} \quad (4.36)$$

Secondly it is straightforward to show

$$P_{NM}\left(-\frac{1}{2\pi^2}\left[\int_0^\pi \tau(w)^2\, dw\right]\left[\int_0^s\int_0^\pi\int_y^x \theta(w)\, dw\, dy\, dx\right]\right)$$

$$= \left(\frac{a^2+b^2}{4}\right)\left(\frac{a}{N^2}\sin Ns + \frac{b}{M^2}\sin Ms\right) \qquad \text{for all } M, N\,. \tag{4.37}$$

Lastly an involved calculation shows that

$$P_{NM}\left(\frac{1}{2\pi}\int_0^s\int_0^\pi\int_y^x (\tau(x) + \tau(y) + \tau(w))^2\theta(w)\, dw\, dy\, dx\right)$$

$$= -\left(\frac{3a^3}{8N^2} + \frac{(5N+3M)ab^2}{4N^2(N+M)}\right)\sin Ns$$

$$-\left(\frac{3b^3}{8M^2} + \frac{(3N+5M)a^2b}{4M^2(N+M)}\right)\sin Ms, \qquad M \neq 3N\,,$$

while

$$P_{NM}\left(\frac{1}{2\pi}\int_0^s\int_0^\pi\int_y^x (\tau(x) + \tau(y) + \tau(w))^2\theta(w)\, dw\, dy\, dx\right)$$

$$= -\left(\frac{3a^3}{8N^2} + \frac{7a^2b}{24N^2} + \frac{7ab^2}{8N^2}\right)\sin Ns$$

$$-\left(\frac{b^3}{24N^2} + \frac{7a^3}{72N^2} + \frac{a^2b}{8N^2}\right)\sin 3Ns\,, \qquad M = 3N.$$

Therefore

$$P_{NM}C_2(e) = \begin{cases} -\dfrac{(M+3N)}{4N^2(M+N)}ab^2\sin Ns - \dfrac{(3M+N)}{4M^2(M+N)}a^2b\sin Ms, & M \neq 3N \\[2ex] -\left(\dfrac{3ab^2}{8N^2} + \dfrac{5a^2b}{12N^2}\right)\sin Ns - \left(\dfrac{11a^3}{108N^2} + \dfrac{5a^2b}{72N^2}\right)\sin 3Ns, & M = 3N. \end{cases}$$

4.8 The bifurcation equations in algebraic form

In order to distinguish between the various cases it is convenient to let

$$\delta_{NM} = 0 \text{ if } M = 2N \text{ and } \delta_{NM} = 1 \text{ if } M \neq 2N$$

$$\varepsilon_{NM} = 1 \text{ if } M = 3N \text{ and } \varepsilon_{NM} = 0 \text{ if } M \neq 3N.$$

Then the bifurcation equation ($\mathcal{B}$) may be written as a pair of algebraic equations as follows:

$$\begin{aligned}
\beta a = {} & \frac{\alpha a}{N} - \frac{3ab}{4N^2}(1 - \delta_{NM}) + \delta_{NM}\frac{9Ma^3}{8N^2(2N-M)} \\
& + \frac{ab^2}{4}\left(\frac{(2M+N)(M^2+4MN+N^2)}{M^2N^2(M+N)} - \delta_{NM}\frac{(2M-N)(M+N)}{MN^2(M-2N)}\right) \\
& - \varepsilon_{NM}\frac{39a^2b}{8N^2} + \frac{(N+M)}{8MN^2}(a^3+2ab^2) \\
& + \varepsilon_{NM}\frac{a^2b}{6N^2} - \left(\frac{(M+3N)ab^2}{4N^2(M+N)} + \varepsilon_{NM}\frac{5a^2b}{12N^2}\right) + h(\alpha,\beta,a,b)
\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}
\beta b = {} & \frac{\alpha b}{M} - \frac{3a^2}{8N^2}(1 - \delta_{NM}) + \frac{9Nb^3}{8M^2(2M-N)} \\
& + \frac{a^2b}{4}\left(\frac{(M+2N)(M^2+4MN+N^2)}{M^2N^2(M+N)} - \delta_{NM}\frac{(M^2+2MN-2N^2)(M+N)}{M^2N^2(M-2N)}\right) \\
& - \varepsilon_{NM}\frac{13a^3}{8N^2} + \frac{(N+M)}{8M^2N}(b^3 + 2a^2b) \\
& + \varepsilon_{NM}\frac{a^3}{54N^2} - \left(\frac{(3M+N)a^2b}{4M^2(M+N)} + \varepsilon_{NM}\frac{11a^3}{108N^2}\right) + k(\alpha,\beta,a,b)\,.
\end{aligned} \tag{4.39}$$

These may be written in the abbreviated form

$$\beta a = f_1(\alpha,\beta,a,b) \tag{B1}$$

$$\beta b = f_2(\alpha,\beta,a,b) \tag{B2}$$

where

$$f_1(\alpha,\beta,a,b) = a_1\alpha a + b_1 ab + pa^3 + ta^2b + rab^2 + h(\alpha,\beta,a,b),$$

$$f_2(\alpha,\beta,a,b) = a_2\alpha b + b_2 a^2 + ua^3 + qa^2b + sb^3 + k(\alpha,\beta,a,b)$$

and $|h| + |k| = O\,(|(\alpha,\beta,a,b)|^4)$ as $(\alpha,\beta,a,b) \to 0$.

Here a_1, a_2, etc. are given as functions of M and N by the formulae

$$a_1 = \frac{1}{N}\,; \qquad a_2 = \frac{1}{M}\,;$$

$$b_1 = -\frac{3}{4N^2}(1-\delta_{NM}); \qquad b_2 = -\frac{3}{8N^2}(1-\delta_{NM})\,;$$

$$p = \frac{1}{8N^2}\left(\frac{9\delta_{NM}N}{2N-M} + \frac{N+M}{M}\right); \qquad u = -\varepsilon_{NM}\frac{41}{24N^2}\,;$$

$$t = -\varepsilon_{NM}\frac{41}{8N^2}\,; \qquad q = \left(\frac{M^2+6MN+2N^2}{4M^2N^2} - \delta_{NM}\frac{(M^2+2MN-2N^2)(M+N)}{4M^2N^2(M-2N)}\right);$$

$$r = \left(\frac{2M^2+6MN+N^2}{4M^2N^2} - \delta_{NM}\frac{(2M-N)(M+N)}{4MN^2(M-2N)}\right); \qquad s = \frac{8N^2+MN+2M^2}{8NM^2(2M-N)}\,.$$

Now is a good time to make a note of the signs of certain combinations of these coefficients. The purpose of these calculations may not be immediately obvious but will become so in §5.3. Suppose $M \neq kN$ for $k = 2$ or 3 (and recall $M > N$). Then

$$a_1 - a_2 = \frac{M-N}{NM} > 0 \text{ for all M and N,} \tag{4.40}$$

$$a_1 q - a_2 p = \frac{10M+N}{8M^2N^2} \quad \text{for all M and N,} \tag{4.41}$$

$$a_1 s - a_2 r = \frac{2M^4-7M^3N+48M^2N^2-28MN^3-4N^4}{8M^3N^2(M-2N)(2M-N)} \tag{4.42}$$

$$\begin{cases} > 0 & \text{if } M > 2N, \\ < 0 & \text{if } M < 2N \ . \end{cases}$$

(To see that the numerator is always positive write it as

$$2M^2\left(M - \frac{7N}{4}\right)^2 + N^2(4(M^2-N^2) + 37\tfrac{7}{8}M^2 - 28MN).)$$

Also

$$ps - qr = \frac{16(N^6-M^6)+18M^5N+33M^4N^2-720M^3N^3+68M^2N^4+160MN^5}{64M^4N^3(M-2N)^2(2M-N)} < 0 \text{ for all M, N.} \tag{4.43}$$

(To see that the numerator is negative write $M = N+k$, it then becomes

$$-425N^6 - 1706N^5k - 1954N^4k^2 - 728N^3k^3 - 117N^2k^4 - 78Nk^5 - 16k^6.)$$

In particular

$$\frac{s-r}{a_1s-a_2r} - \frac{q-p}{a_1q-a_2p} = \frac{(a_1-a_2)(ps-qr)}{(a_1s-a_2r)(a_1q-a_2p)}$$

$$\begin{cases} > 0 & \text{if} \quad N < M < 2N \\ < 0 & \text{if} \quad M > 2N. \end{cases} \tag{4.44}$$

4.9 Further Properties of the Bifurcation Equations when M = kN

It was proved in §4.6 that

$$f_1(\alpha,\beta,0,b) = 0 \qquad \text{for all} \quad (\alpha,\beta,0,b) \in U$$

and that, provided $M \neq kN$, $k \in \mathbb{N}$, that

$$f_2(\alpha,\beta,a,0) = 0 \qquad \text{for all} \quad (\alpha,\beta,a,0) \in U.$$

It was also pointed out that if $M = kN$ the proof of the second result breaks down and indeed, we suspect that it is false in this case.

When $k = 2$ and $k = 3$ an inspection of (4.39) shows that $f_2(\alpha,\beta,a,b)$ contains a term in a^2 when $k = 2$ and a^3 when $k = 3$ and hence in these cases

$$f_2(\alpha,\beta,a,0) \neq 0$$

for all $(\alpha,\beta,a,0) \in U$.

In this section we shall present the calculations which show that this result is also true when $k = 4,5$. Specifically it will be shown that the series expansion of $G(\nu_{NM}, a \sin Ns + z(0,0,a,0))$ contains a term in $a^k \sin kNs$, $k = 4,5$; or more precisely that, if $\overline{P}_k$ is the projection of X_1 onto sp{sin kNs} , then

$$\frac{d^k}{da^k}\, \overline{P}_k G(\nu_{NM}, a \sin Ns + z(0,0,a,0))\big|_{a=0} \neq 0 \ , \ k = 4,5.$$

First write $G(\nu_{NM},\theta)$ as

$$\nu_{NM}A\theta + B\theta + Q(\theta) + \nu_{NM}C_1(\theta) + C_2(\theta) + \frac{F^{(iv)}(\theta)}{4!} + \frac{F^{(v)}(\theta)}{5!} + H(\theta) \tag{4.45}$$

where $F^{(iv)}(\theta)$ and $F^{(v)}(\theta)$ denote respectively the fourth and fifth order Fréchet derivatives with respect to θ of $G(\nu_{NM}, \theta)$ evaluated at $\theta = 0$ and $H(\theta) = 0(||\theta||^6)$ as $||\theta|| \to 0$.

The next step is to find the coefficient of $a^2\sin 2Ns$, $a^3\sin 3Ns$ and, if $k = 5$, $a^4\sin 4Ns$ in the series expansion of $z(0,0,a,0)$. In other words it is required to evaluate

$$\frac{1}{t!}\frac{d^t}{da^t}\overline{P}_t z(0,0,a,0)\Big|_{a=0} \equiv A_t \qquad \text{for } t = 2,3 \tag{4.46}$$

and if $k = 5$

$$\frac{1}{4!}\frac{d^4}{da^4}\overline{P}_4 z(0,0,a,0)\Big|_{a=0} \equiv A_4 \quad . \tag{4.47}$$

(If $k = 4$ this latter term is zero since $\sin 4Ns \in E_{N4N}$ and $z(\alpha,\beta,a,b) \in F_{N4N}$. Of course, A_t is a function of k, but this has been suppressed for convenience of notation.) It is already known from (4.33) that

$$A_2 = \frac{3k}{2(k-2)} \quad . \tag{4.48}$$

Differentiating (4.17) three times yields that

$$A_3 = (\gamma_{NkN}I - \nu_{NkN}A - B)^{-1}\circ\overline{P}_3\{2q(\sin Ns, A_2\sin 2Ns) + \nu_{NkN}C_1(\sin Ns) + C_2(\sin Ns)\} \tag{4.49}$$

and on evaluating the right-hand side there results that

$$A_3 = \frac{9kN^2}{2(3-k)}\left(\frac{-13k}{24(k-2)N^2} + \frac{k+1}{72kN^2} - \frac{11}{108N^2}\right)$$

$$\begin{cases} = \dfrac{1009}{48} & \text{if } k = 4, \\[2ex] = \dfrac{1067}{96} & \text{if } k = 5. \end{cases} \tag{4.50}$$

If k = 5 a further differentiation of (4.17) yields that

$$A_4 = (\gamma_{N5N}I - \nu_{N5N}A - B)^{-1}\circ\bar{P}_4\circ\{Q(A_2\sin 2Ns) + 2q(\sin Ns,\ A_3\sin 3Ns)$$

$$+ 3\nu_{N5N}\tilde{C}_1(\sin Ns,\ \sin Ns,\ A_2\sin 2Ns) + 3\tilde{C}_2(\sin Ns,\ \sin Ns,\ A_2\sin 2Ns)$$

$$+ \frac{F^{(iv)}}{4!}(\sin Ns)\} \tag{4.51}$$

where $\tilde{C}_i(\theta_1,\theta_2,\theta_3)$, i = 1, 2 are the tri-linear functions such that $\nu_{N5N}6\tilde{C}_1 + 6\tilde{C}_2$ is the third-order Fréchet derivative of $G(\nu_{N5N},\theta)$ with respect to θ evaluated at $\theta = 0$. Hence $\tilde{C}_i(\theta,\theta,\theta) = C_i(\theta)$. A rather lengthy calculation then yields that

$$A_4 = \frac{80N^2}{3}\left(\frac{75}{128N^2} + \frac{1529}{576N^2} - \frac{3}{32N^2} + \frac{85}{192N^2} + \frac{23}{1152N^2}\right) = \frac{1155}{12}\ . \tag{4.52}$$

From now on we shall consider the cases k = 4 and k = 5 separately.

<u>k=4</u>

It is required to evaluate

$$\frac{1}{4!}\frac{d^4}{da^4}\bar{P}_4G(\nu_{N4N},\ a\sin Ns + z(0,0,a,0))\big|_{a=0}\ . \tag{4.53}$$

Obviously this is the same as

$$\frac{1}{4!}\frac{d^4}{da^4}\,\overline{P}_4 G(\nu_{N4N}, a \sin Ns + A_2 a^2 \sin 2Ns + A_3 a^3 \sin 3Ns)\big|_{a=0} \quad ,$$

which is equal to

$$\begin{aligned}\overline{P}_4\{&Q(A_2 \sin 2Ns) + 2q(\sin Ns, A_3 \sin 3Ns)\\ &+3\nu_{N4N}\tilde{C}_1(\sin Ns, \sin Ns, A_2 \sin 2Ns)\\ &+3\tilde{C}_2(\sin Ns, \sin Ns, A_2 \sin 2Ns)\\ &+\frac{F^{(iv)}}{4!}(\sin Ns)\} \quad .\end{aligned}$$

Then a series of calculations shows that

$$\overline{P}_4 Q(\sin 2Ns) = -\frac{3}{32N^2}\sin 4Ns$$

$$\overline{P}_4 q(\sin Ns, \sin 3Ns) = -\frac{11}{96N^2}\sin 4Ns$$

$$\overline{P}_4\tilde{C}_1(\sin Ns, \sin Ns, \sin 2Ns) = \frac{1}{96N^2}\sin 4Ns$$

$$\overline{P}_4\tilde{C}_2(\sin Ns, \sin Ns, \sin 2Ns) = \frac{17}{288N^2}\sin 4Ns$$

$$\overline{P}_4\frac{F^{(iv)}}{4!}(\sin Ns) = -\frac{23}{1152N^2}\sin 4Ns.$$

So (4.53) is equal to

$$- \frac{3A_2^2}{32N^2} - \frac{11A_3}{48N^2} + \frac{5A_2}{128N^2} + \frac{17A_2}{96N^2} - \frac{23}{1152N^2}$$

$$= - \frac{27}{32N^2} - \frac{11\times 1009}{48^2N^2} + \frac{15}{128N^2} + \frac{17\times 3}{96N^2} - \frac{23}{1152N^2}$$

$$= - \frac{3865}{768N^2} \quad .$$

<u>k=5</u>

The quantity to be evaluated is in this case

$$\frac{1}{5!}\frac{d^5}{da^5}\overline{P}_5 G(\nu_{N5N}, a \sin Ns + A_2 a^2 \sin 2Ns + A_3 a^3 \sin 3Ns + A_4 a^4 \sin 4Ns)\big|_{a=0} \tag{4.54}$$

which is equal to

$$\overline{P}_5\{2q(\sin Ns,\ A_4 \sin 4Ns) + 2q(A_2 \sin 2Ns,\ A_3 \sin 3Ns)$$

$$+ 3\nu_{N5N}\tilde{C}_1(\sin Ns,\ \sin Ns,\ A_3 \sin 3Ns) + 3\nu_{N5N}\tilde{C}_1(\sin Ns,\ A_2 \sin 2Ns,\ A_2 \sin 2Ns)$$

$$+ 3\tilde{C}_2(\sin Ns,\ \sin Ns,\ A_3 \sin 3Ns) + 3\tilde{C}_2(\sin Ns,\ A_2 \sin 2Ns,\ A_2 \sin 2Ns)$$

$$+ \frac{\tilde{F}^{(iv)}}{3!}(\sin Ns,\ \sin Ns,\ \sin Ns,\ A_2 \sin 2Ns) + \frac{F^{(v)}}{5!}(\sin Ns)\}$$

where $\tilde{F}^{(iv)}(\theta_1,\theta_2,\theta_3,\theta_4)$ is the 4-linear function such that $\tilde{F}^{(iv)}(\theta,\theta,\theta,\theta)$ is the fourth order Fréchet derivative of $G(\nu_{N5N},\theta)$ with respect to θ, evaluated at $\theta = 0$. Hence $\tilde{F}^{(iv)}(\theta,\theta,\theta,\theta) = F^{(iv)}(\theta)$.

Then a series of calculation yields that

$$\bar{P}_5 q(\sin Ns, \sin 4Ns) = -\frac{33}{400N^2}$$

$$\bar{P}_5 q(\sin 2Ns, \sin 3Ns) = -\frac{37}{600N^2}$$

$$\bar{P}_5 \tilde{C}_1(\sin Ns, \sin Ns, \sin 3Ns) = \frac{1}{120N}$$

$$\bar{P}_5 \tilde{C}_1(\sin Ns, \sin 2Ns, \sin 2Ns) = \frac{1}{120N}$$

$$\bar{P}_5 \tilde{C}_2(\sin Ns, \sin Ns, \sin 3Ns) = -\frac{19}{450N^2}$$

$$\bar{P}_5 \tilde{C}_2(\sin Ns, \sin 2Ns, \sin 2Ns) = -\frac{17}{225N^2}$$

$$\bar{P}_5 \frac{\tilde{F}^{(iv)}}{3!}(\sin Ns, \sin Ns, \sin Ns, \sin 2Ns) = -\frac{419}{14400N^2}$$

$$\bar{P}_5 \frac{F^{(v)}}{5!}(\sin Ns) = -\frac{97}{21600N^2}$$

Hence (4.54) equals

$$-\frac{33A_4}{200N^2} - \frac{37A_2A_3}{300N^2} + \frac{6}{5N}\left(\frac{A_3}{40N} + \frac{A_2^2}{40N}\right)$$

$$-\frac{19A_3}{150N^2} - \frac{17A_2^2}{75N^2} - \frac{419A_2}{14400N^2} - \frac{97}{21600N^2}$$

$$= - \frac{33A_4}{200N^2} - \frac{37A_2A_3}{300N^2} - \frac{29A_3}{300N^2} - \frac{59A_2^2}{300N^2}$$

$$- \frac{419A_2}{14400N^2} - \frac{97}{21600N^2}$$

$$= - \frac{33\times 1155}{200\times 12N^2} - \frac{37\times 5\times 1067}{300\times 2\times 96N^2} - \frac{29\times 1067}{300\times 96N^2}$$

$$- \frac{59\times 25}{300\times 4N^2} - \frac{419\times 5}{14400\times 2N^2} - \frac{97}{21600N^2}$$

$$= - \frac{3747869}{172800N^2} \ .$$

Hence we have proved that for $M = kN$ and $k = 2,3,4,5$

$$f_2(\alpha,\beta,a,0) \neq 0 \qquad \text{for all} \qquad (\alpha,\beta,a,0) \in U.$$

As has been remarked, we believe this to be true in the general case, although a proof is lacking.

CHAPTER V

THE BIFURCATION ANALYSIS

5.1 Introduction

We begin with a brief recap of what has been done so far. We wish to describe the solution set of

$$(\gamma_{NM}+\beta)\theta = G(\nu_{NM}+\alpha,\ \theta) \qquad (N)$$

for (α,β,θ) in a neighbourhood of the origin in $\mathbb{R}\times\mathbb{R}\times X_1$. Here γ_{NM} is a critical value of the surface tension with corresponding double eigenvalue ν_{NM} whose eigenspace is spanned by sin Ns and sin Ms. Since the surface tension is a naturally pre-assigned parameter what we shall actually do is to study the solution set (α,θ) of (N) in a neighbourhood of $(0,0)$ for various fixed values of β. First for $\beta = 0$ in which case the only possible primary bifurcation point is the double eigenvalue at $\alpha = 0$; secondly for $\beta \neq 0$ in which case the double eigenvalue is perturbed into two simple eigenvalues at $\alpha = N\beta$ and $\alpha = M\beta$. We wish to discover the existence and multiplicity of any primary curves which bifurcate from these points and also whether there are any secondary bifurcation points on these curves.

To this end we have reduced problem (N) to the finite dimensional form (B) which consists of two polynomial equations in four unknowns. General results on bifurcation when a double eigenvalue splits, under perturbation, into two simple eigenvalues have been obtained by Shearer (1978, 1980) and we shall make use of some of his general results. Bifurcation equations of the type studied here have also been considered by Golubitsky & Schaeffer (1979) and Chow & Hale (1982).

The results on symmetry and invariant subspaces noted in Chapter IV will be essential for the bifurcation analysis. Indeed it should be emphasised that there is no guarantee that our analysis locates all bifurcation points, but what is certain is that it identifies all those which arise as a consequence of the symmetries inherent in the problem. There is no obvious reason why there should be any others.

The symmetry observations will also be needed in Chapter VI to identify distinct solutions θ of (N) with the same capillary-gravity wave.

The notion of a primary bifurcation point was recalled in §4.1, now recall that of a secondary bifurcation point.

DEFINITION. *For fixed* β *let* $\Gamma_\beta = \{(\alpha,0) : \alpha \in \mathbb{R}\} \subset \mathbb{R}\times X_1$ *denote the line of trivial solutions of* (N). *Let* $C = \{(\alpha(t),\theta(t)) : t \in (-\delta,\delta)\}$ *denote a primary branch of solutions which bifurcates from* Γ_β *at* $(\alpha_\beta,0)$. *If* $\mathcal{D} = \{(\hat{\alpha}(t),\hat{\theta}(t)) : t \in (-\delta,\delta)\}$ *is a branch of solutions such that* $C \cap \mathcal{D} = \{(\hat{\alpha},\hat{\theta})\} \subset C \setminus \{(\alpha_\beta,0)\}$ *then* $(\hat{\alpha},\hat{\theta})$ *is said to be a secondary bifurcation point and* $\mathcal{D}$ *a secondary bifurcation branch or curve.*

DEFINITION. *Suppose* $(\alpha_\beta,0)$ *is a primary bifurcation point and* $C = \{(\alpha(t),\theta(t)) : t\in(-\delta,\delta)\}$ *is a primary bifurcation curve passing through* $(\alpha_\beta,0)$ *at* $t = 0$. *If* $\alpha(t) < \alpha_\beta$ $(\alpha(t) > \alpha_\beta)$ *for all* $t \in (-\delta,\delta) \setminus \{0\}$ *then* C *is said to bifurcate sub-(super-) critically and if* $t(\alpha(t)-\alpha_\beta) < 0$ *or* $t(\alpha(t)-\alpha_\beta) > 0$, $t \in (-\delta,\delta) \setminus \{0\}$ *then* C *is said to bifurcate transcritically.*

5.2 The Methods

To obtain solutions (α,β,θ) of (N) we first determine solutions (α,β,a,b) of $(\mathcal{B})$ and then obtain the corresponding solutions $(\alpha,\beta,\ a \sin Ns + b \sin Ms + z(\alpha,\beta,a,b))$ of (N). The main problem is to deduce the nature of small solutions of the bifurcation equations from the knowledge of their power series expansion up to cubic order. Our tools in this task will be the Implicit Function Theorem (see §4.5) and the Blowing-Up Lemma which enables us to determine, up to homeomorphism, the solution set of a mapping in a neighbourhood of the origin in terms of that of its truncated Taylor series. This lemma will be used in a form given by Shearer (1978) who generalised a result of Buchner, Marsden and Schecter (1983). We need some definitions and notation. Suppose m and n are positive integers with $m \geq n+1$ and $\underline{m}$ and $\underline{q}$ denote m tuples of positive and non-negative integers respectively, while $\underline{n}$ denotes an n-tuple of positive integers. Let $|\underline{q}| = \sum_{i=1}^{m} q_i$ if $\underline{q} = \{q_i\}_{i=1}^{m}$ and if $f : \mathbb{R}^m \to \mathbb{R}$ is smooth let

$$f_{\underline{q}} = \frac{\partial^{|\underline{q}|} f}{\partial x_1^{q_1} \partial x_2^{q_2} \cdots \partial x_m^{q_m}}$$

THE BLOWING-UP LEMMA. <u>Suppose</u> $\underline{g} : \mathbb{R}^m \to \mathbb{R}^n$ <u>is</u> C^∞ <u>and is given by</u> $\underline{g} = (g_1, g_2, \ldots, g_n)$, <u>and that there exists</u> $\underline{m}$ <u>and</u> $\underline{n}$ <u>as above with the property that for each</u> i, $1 \leq i \leq n$

$$(g_i)_{\underline{q}}(0) = 0 \qquad \underline{\text{if}} \quad \underline{m}.\underline{q} = \sum_{j=1}^{m} m_j q_j \leq n_i - 1 \ .$$

<u>Define the Taylor polynomial</u> $\underline{f} = (f_1, f_2, \ldots, f_n)$ <u>by</u>

$$f_i(x) = \sum_{\underline{m}.\underline{q}=n_i} \{(q_1! q_2! \ldots q_n!)^{-1} (g_i)_{\underline{q}}(0)\} x_1^{q_1} x_2^{q_2} \ldots x_m^{q_m} \ .$$

If $\underline{f}(x) = 0$ and $x \neq 0$ imply that $D\underline{f}(x) : \mathbb{R}^m \to \mathbb{R}^n$ has rank n, then there exist neighbourhoods U and V of the origin in $\mathbb{R}^m$ and a homeomorphism ϕ from $\underline{g}^{-1}(0) \cap U$ onto $\underline{f}^{-1}(0) \cap V$ such that $\phi(0) = 0$. Moreover $(\underline{g}^{-1}(0) \cap U) \setminus \{0\}$ is a C^∞ submanifold of $\mathbb{R}^m$ and ϕ is a diffeomorphism of it onto $(\underline{f}^{-1}(0) \cap V) \setminus \{0\}$.

Remark. It should be noted that the Blowing-Up Lemma on its own is not enough to determine the solution set of $(\mathcal{B})$ for fixed β. The lemma gives no information about the homeomorphism (except $\phi(0) = 0$) and thus although the solution set in (α,β,a,b) space may be left qualitatively unchanged, that for fixed β may be substantially different.

From now on we consider various cases separately. The two main cases are $M = kN$ and $M \neq kN$, each containing a number of sub-cases.

5.3 The Case $M \neq kN$

In this case the formulae of §4.8 dictate that the bifurcation equations are

$$\beta a = \frac{\alpha a}{N} + \frac{(8M^2+MN+2N^2)}{8MN^2(2N-M)} a^3 + \frac{(M^2-10MN-2N^2)}{4M^2N(M-2N)} ab^2 + h(\alpha,\beta,a,b) \tag{5.1a}$$

and

$$\beta b = \frac{\alpha b}{M} + \frac{(8N^2+MN+2M^2)}{8M^2N(2M-N)} b^3 + \frac{(M^2-10MN-2N^2)}{4M^2N(M-2N)} a^2b + k(\alpha,\beta,a,b) \tag{5.1b}$$

where h and k are real-analytic and

$$|(h,k)| = O(|(\alpha,\beta,a,b)|^4) \qquad \text{as } (\alpha,\beta,\ a,\ b) \to 0.$$

The symmetry observations of the previous chapter coupled with the real-analyticity of h and k means that $(h,k) = (ah_1,\ bk_1)$ where

$$|(h_1,k_1)| = O(|(\alpha,\beta,a,b)|^3) \qquad \text{as } (\alpha,\beta,\ a,\ b) \to 0.$$

We noted in §4.6 that the problem (N) may be posed in X_N or X_M and that for all values of $\beta, \alpha = N\beta$ and $\alpha = M\beta$ are simple eigenvalues in the appropriate context.

The next result is a consequence of looking at the problem from this point of view. It is Theorem 2.1 of Shearer (1978, 1980) and is a two parameter version of Theorem 4.1 of this thesis. For $d > 0$, let $R_d = (-d,d)\times(-d,d)$.

THEOREM 5.1. *Suppose* $M \neq kN$ *for all* $k \in \mathbb{N}$. *Then there exist* $d > 0$ *and continuous functions* α_M, α_N *which each map* R_d *into* $\mathbb{R}$ *and satisfy*

$$\alpha_M(\beta,0) = M\beta \quad \text{and} \quad \alpha_N(\beta,0) = N\beta,$$

such that for fixed $\beta \in (-d,d)$ *the curves*

$$C_\beta^N = \{(\alpha_N(\beta,a),\ a \sin Ns + z(\alpha_N,\beta,a,0)) : |a| < d\}$$

and

$$C_\beta^M = \{(\alpha_M(\beta,b),\ b \sin Ms + z(\alpha_M,\beta,0,b)) : |b| < d\}$$

are primary curves of solutions of (N) *which bifurcate at* $\alpha = N\beta$ *and* $\alpha = M\beta$ *respectively.*

Moreover there is a $\delta > 0$ *such that if* $(\alpha,\beta,\theta) \in \mathbb{R}\times\mathbb{R}\times X_N$ *is a solution of* (N) *satisfying* $|\alpha| < \delta$, $|\beta| < d$ *and* $||\theta|| < \delta$ *then either* $\theta \equiv 0$ *or* $(\alpha,\theta) \in C_\beta^N$. *Similarly if* $(\alpha,\beta,\theta) \in \mathbb{R}\times\mathbb{R}\times X_M$ *is a solution of* (N) *satisfying the same bounds then either* $\theta \equiv 0$ *or* $(\alpha,\theta) \in C_\beta^M$.

To facilitate the understanding of the solution set of (N) in a neighbourhood of the origin in $\mathbb{R}\times\mathbb{R}\times X_1$ it is helpful to determine the approximate form of C_β^N and C_β^M. A straightforward calculation using (5.1a) yields that

$$C_\beta^N = \left\{ (N\beta - \frac{(8M^2+MN+2N^2)}{8MN(2N-M)}a^2,\ a \sin Ns) + O(a^2(|(a,\beta)|,\ 1)) : |a| < d \right\}. \tag{5.2}$$

(Recall from (4.20) that $z(\alpha,\beta,a,b) = O(|(a,b)|^2)$ as $(\alpha,\beta,a,b) \to 0$.)

An analogous calculation using (5.1b) yields that

$$\mathcal{C}_\beta^M = \left\{ (M\beta - \frac{(8N^2+MN+2M^2)}{8MN(2M-N)} b^2,\ b \sin Ms) + O(b^2(|(b,\beta)|,\ 1)) \ :\ |b| < d \right\}. \tag{5.3}$$

Let $\mathcal{C}_\beta^N = {}^+\mathcal{C}_\beta^N \cup {}^-\mathcal{C}_\beta^N \cup \{(N\beta,0)\}$ where ${}^\pm\mathcal{C}_\beta^N$ denotes the component of $\mathcal{C}_\beta^N \setminus \{(N\beta,0)\}$ along which $\pm a > 0$. Let ${}^\pm\mathcal{C}_\beta^M$ be defined similarly. The formulae (5.2) and (5.3) then dictate that $\mathcal{C}_\beta^M$ is always a sub-critical pitchfork bifurcation while $\mathcal{C}_\beta^N$ is a sub-critical pitchfork bifurcation if $M < 2N$ and a super-critical pitchfork bifurcation if $M > 2N$. We will see later that both branches of a pitchfork correspond to the same set of capillary-gravity waves although they are different sets of solutions of (N).

Theorem 5.1 accounts for all solutions of (N) in a neighbourhood of the origin in $\mathbb{R}\times\mathbb{R}\times(X_N \cup X_M)$. Consequently no solutions are lost if (5.1a) is divided by a and (5.1b) is divided by b. The resulting equations may be written in abbreviated form (see §4.8) as

$$\beta = a_1\alpha + pa^2 + rb^2 + h_1(\alpha,\beta,a,b) \quad , \tag{5.4a}$$

$$\beta = a_2\alpha + qa^2 + sb^2 + k_1(\alpha,\beta,a,b) \quad . \tag{5.4b}$$

The first step in the analysis of these equations is to apply the Blowing-Up Lemma to (5.4). If we take $m = 4$, $n = 2$, $\underline{m} = (2,2,1,1)$, $\underline{n} = (2,2)$, the Taylor polynomial equation which results is

$$\begin{pmatrix} a_1\alpha - \beta + pa^2 + rb^2 \\ \\ a_2\alpha - \beta + qa^2 + sb^2 \end{pmatrix} = \underline{f}(\alpha,\beta,a,b) = 0 \tag{5.5}$$

and $D\underline{f}$ is

$$\begin{pmatrix} a_1, & -1, & 2pa, & 2rb \\ a_2, & -1, & 2qa, & 2sb \end{pmatrix} \quad .$$

If $\underline{f}(\alpha,\beta,a,b) = 0$ and $(\alpha,\beta,a,b) \neq (0,0,0,0)$ it is easy to see that the rank of $D\underline{f}$ is two since $a_2 - a_1 \neq 0$. Hence there is a smooth diffeomorphism between the solutions of (5.4) in a punctured neighbourhood of the origin in $\mathbb{R}^4$ and the solution set of (5.5) in another punctured neighbourhood of the origin in $\mathbb{R}^4$. However to obtain a precise description of this solution set for fixed β we must investigate further. We shall consider the cases $\beta = 0$, $\beta \neq 0$ separately.

<u>5.3(a) $\beta \neq 0$</u>

Since $\alpha = M\beta$ and $\alpha = N\beta$ are the only eigenvalues of the linearised problem near $\alpha = 0$ and each is a simple eigenvalue, we have found all the primary bifurcation curves in a neighbourhood of the origin. Now the object is to determine any secondary bifurcations which occur on $C_\beta^M \cup C_\beta^N$. The procedure to find these is as follows. Suppose that for β fixed in the interval $(-d,d)$, there exists a sequence $\{(\alpha_m,a_m,b_m)\}$ of solutions of (5.4) such that $b_m \neq 0$ for each m and $|\alpha_m| < \delta$, $||a_m \sin Ns + b_m \sin Ms + z(\alpha_m,\beta,a_m,b_m)|| < \delta$, and that $(\alpha_m,a_m,b_m) \to (\alpha_o,a_o,0)$ as $m \to \infty$, $(a_o \neq 0)$. Then $\theta_o = a_o \sin Ns + z(\alpha_o,\beta,a_o,0) \in X_N$ and $(\alpha_o,\beta,\theta_o)$ is a solution of (N). Therefore by Theorem 5.1 $(\alpha_o,\theta_o) \in C_\beta^N$. However $\theta_m = a_m \sin Ns + b_m \sin Ms + z(\alpha_m,\beta,a_m,b_m) \notin X_N$ for any $m \geq 1$, although $(\alpha_m,\beta,\theta_m)$ is a solution of (N). Since $(\alpha_m,\theta_m) \to (\alpha_o,\theta_o) \in C_\beta^N$ where $\theta_o \not\equiv 0$ and $(\alpha_m,\theta_m) \notin C_\beta^N$ for any $m \geq 1$, (α_o,θ_o)

must be a secondary bifurcation point on C^N_β. The same argument with the roles of a and b interchanged demonstrates the existence of secondary bifurcation points on C^M_β.

Thus secondary bifurcation points correspond to solutions of (5.4) for which ab = 0. The next result, based on Lemma 5.6 of Shearer (1978), gives "a-priori" bounds for such solutions.

LEMMA 5.2. <u>For all sufficiently small solutions</u> (α,β,a,b) <u>of</u> (5.4), <u>there exist positive constants</u> c_1 <u>and</u> c_2 <u>such that</u>

$$c_1|(\alpha,\beta)| \le |(a,b)|^2 \le c_2|(\alpha,\beta)|. \tag{5.6a}$$

<u>Further, for any solution with</u> ab = 0 <u>there exist positive constants</u> d_1, d_2 <u>such that</u>

$$d_1|\beta| \le |\alpha| \le d_2|\beta|. \tag{5.6b}$$

<u>Proof</u>. Since (5.4) may be written

$$(\alpha,\beta) = L(a^2,b^2) + (h_1,k_1)$$

where, as a calculation easily verifies, L is an invertible linear operator; (5.6a) follows at once. We shall prove the right-hand side of (5.6b) for the case b = 0, the other situations being similar.

If the estimate did not hold then there would be a sequence of solutions of (5.4) such that $(\alpha_n, \beta_n, a_n) \to (0,0,0)$ and that $\frac{\beta_n}{\alpha_n} \to 0$ as $n \to \infty$. Then substituting the sequence into (5.4) and dividing by α_n we have

$$\frac{\beta_n}{\alpha_n} = a_1 + p\frac{a_n^2}{\alpha_n} + \frac{h_1}{\alpha_n} \quad ,$$

$$\frac{\beta_n}{\alpha_n} = a_2 + q\,\frac{a_n^2}{\alpha_n} + \frac{k_1}{\alpha_n} \quad .$$

Now by (5.6a), $\{a_n^2/\alpha_n\}$ is a bounded sequence and so has a convergent subsequence (labelled the same) with limit a^*. Also $\alpha_n^{-1}h_1(\alpha_n,\beta_n,a_n,0)$ and $\alpha_n^{-1}k_1(\alpha_n,\beta_n,a_n,0)$ both tend to zero as $n \to \infty$ since $|h_1| + |k_1| = O(|(\alpha,\beta,a)|^3)$. So letting $n \to \infty$ in the preceding equations we have

$$a_1 + pa^* = 0, \qquad a_2 + qa^* = 0$$

which is a contradiction.

q.e.d.

Thus all small solutions of (5.4) with $ab = 0$ may be written as

$$(\alpha,\beta,a,b) = (\sigma^2\Lambda,\ \pm\sigma^2,\ \sigma x,\ \sigma y) \tag{5.7}$$

where $\sigma > 0$ is small, Λ, x and y are bounded and $xy = 0$. These correspond to the only possible secondary bifurcation points on $C_\beta^M \cup C_\beta^N$. Making these substitutions yields the equations (on dividing by σ^2)

$$\pm 1 = a_1\Lambda + px^2 + ry^2 + h_\pm(\sigma,\Lambda,x,y) \qquad (5.8a)^\pm$$

$$\pm 1 = a_2\Lambda + qx^2 + sy^2 + k_\pm(\sigma,\Lambda,x,y) \qquad (5.8b)^\pm$$

where $h_{\pm}(\sigma,\Lambda,x,y) = \sigma^{-2}h_1(\sigma^2\Lambda,\pm\sigma^2,\sigma x,\sigma y)$ for $\sigma \neq 0$ and is defined by analytic continuation to be zero at $\sigma = 0$, $k_{\pm}(\sigma,\Lambda,x,y)$ being defined similarly. It should be noted that $h_{\pm}$ and $k_{\pm}$ are both real analytic, $h_{\pm}(\sigma,\Lambda,0,0) = k_{\pm}(\sigma,\Lambda,0,0) = 0$ and that at zero the first derivatives of $h_{\pm}$, $k_{\pm}$ with respect to each of σ, Λ, x, y are zero. The procedure is now to find all solutions of $(5.8)^{\pm}$ with $\sigma = 0$ passing through $xy = 0$. We then appeal to the Implicit Function Theorem to deduce the existence of such solutions of (5.8) when $\sigma > 0$ is small. Finally the 'a priori' bounds enable us to assert that these correspond to solutions of (5.4) passing through points with $ab = 0$ and hence to secondary bifurcation points on $C^M_\beta \cup C^N_\beta$.

When $\sigma = 0$, $(5.8)^{\pm}$ is equivalent to

$$\frac{(ps-qr)}{(a_1s-a_2r)}x^2 + \Lambda = \frac{\pm(r-s)}{(a_1s-a_2r)}$$

$$\frac{(ps-qr)}{(a_2p-a_1q)}y^2 + \Lambda = \frac{\pm(q-p)}{(a_2p-a_1q)} .$$

The formulae (4.40) - (4.44) now indicate that the situation is different according as $M > 2N$ and $M < 2N$. Therefore we consider these cases separately.

Case I. $M > 2N$

In this case $(5.8)^-$ has no solutions whereas $(5.8)^+$ has a closed loop of solutions passing through the four points

$$\Lambda_1 = \frac{r-s}{a_2r-a_1s} \quad , \quad x = 0 \quad , \quad y_I^2 = \frac{a_1-a_2}{a_1s-a_2r}$$

$$\Lambda_2 = \frac{p-q}{a_2p-a_1q} \quad , \quad y = 0 \quad , \quad x_1^2 = \frac{a_1-a_2}{a_1q-a_2p} \quad .$$

(Note that by (4.44) $\Lambda_1 < \Lambda_2$, which fact will be needed later when we determine the direction of secondary bifurcation.)

Without loss of generality suppose this simple closed loop is parameterised by

$$\{(\hat{\Lambda}(t), \hat{x}(t), \hat{y}(t)) : t \in [0, 2\pi]\}$$

which passes through the four distinguished points at $t = k\pi/2$, $k = 0,1,2,3$. Let C denote the Banach space of all continuous 2π-periodic functions $(\Lambda,x,y) : [0, 2\pi] \to \mathbb{R}^3$, and define $F : \mathbb{R}\times C \to C$ by

$$F(\sigma,(\Lambda,x,y)) = \begin{pmatrix} a_1\Lambda+px^2+ry^2+h_+(\sigma,\Lambda,x,y) \\ a_2\Lambda+qx^2+sy^2+k_+(\sigma,\Lambda,x,y) \\ 2(ps-qr)\hat{x}\hat{y}\Lambda+(a_2r-a_1s)x\hat{y}+(a_1q-a_2p)\hat{x}y \end{pmatrix} .$$

Now the determinant of the derivative of F with respect to (Λ,x,y) evaluated at $(\sigma,\Lambda,x,y) = (0,\hat{\Lambda},\hat{x},\hat{y})$ is $2\hat{x}^2(a_1q - a_2p)^2 + 2\hat{y}^2(a_2r - a_1s)^2 + 8\hat{x}^2\hat{y}^2(ps - qr)^2$ which is non-zero everywhere. It is therefore immediate from the Implicit Function Theorem that

$$F(\sigma,(\Lambda,x,y)) = \begin{pmatrix} 1 \\ 1 \\ \hat{x}\hat{y}(2(ps-qr)\hat{\Lambda} + (a_2r-a_1s) + (a_1q - a_2p)) \end{pmatrix}$$

has a unique solution close to $(\hat{\Lambda}, \hat{x}, \hat{y})$ for $\sigma > 0$ sufficiently small. Let it be denoted by $(\hat{\Lambda}_\sigma, \hat{x}_\sigma, \hat{y}_\sigma)$. Clearly $\hat{x}_\sigma$ and $\hat{y}_\sigma$ each have exactly two zeros close to the zeros of x and y. Therefore when $\beta = +\sigma^2$, there is a closed loop of solutions of (5.4) : $(\alpha_\beta, a_\beta, b_\beta) = (\sigma^2\hat{\Lambda}_\sigma, \sigma\hat{x}_\sigma, \sigma\hat{y}_\sigma)$ along which a_β and b_β are each zero exactly twice. Hence there is a closed curve of solutions of $(\mathcal{N})$:
$\mathcal{D}_\beta = \{(\alpha_\beta, \theta_\beta)\} = \{(\alpha_\beta, a_\beta \sin Ns + b_\beta \sin Ms + z(\alpha_\beta, \beta, a_\beta, b_\beta))\} \subset \mathbb{R}\times X_1$ on which there are four secondary bifurcation points: one on each of $^{\pm}C^M_\beta$ and $^{\pm}C^N_\beta$. Note also that at the secondary bifurcation points on C^N_β (i.e. those at which b = 0), $\alpha_\beta \approx \Lambda_2\beta$ while at the secondary bifurcation points on C^M_β, $\alpha_\beta \approx \Lambda_1\beta$.

Then since $\beta > 0$ and $\Lambda_1 < \Lambda_2$, it follows that the secondary bifurcation from C^N_β is sub-critical and that from C^M_β is super-critical. Observe that $\mathcal{D}_\beta$ is divided into four sections by C^N_β and C^M_β, we label these by $^{+}\mathcal{D}^1_\beta$ where $a,b > 0$, $^{-}\mathcal{D}^1_\beta$ where $a > 0$, $b < 0$, $^{+}\mathcal{D}^2_\beta$ where $a < 0$, $b > 0$ and $^{-}\mathcal{D}^2_\beta$ where $a, b < 0$. We shall see in Chaper VI that as a consequence of the symmetry observations of Chapter IV the four solution sets $^{\pm}\mathcal{D}^i_\beta$, $i = 1,2$ correspond to only two distinct solution sets of the capillary-gravity wave problem.

Since $(5.8)^-$ has no solutions it follows that when $\beta < 0$ there is no secondary bifurcation on $C^M_\beta \cup C^N_\beta$.

These conclusions are shown in Figure 1 . A solution (α,θ) where $\theta = a \sin Ns + b \sin Ms + z(\alpha,\beta,a,b)$ is represented on the diagram by the triple (α,a,b). (Note that a and b are the projections of θ onto sin Ns and sin Ms respectively.)

<u>Case II</u>. $N < M < 2N$

In this case both $(5.8)^+$ and $(5.8)^-$ have non-trivial solutions.

In the case of $(5.8)^+$ there is a curve passing through each of the two points

$$\Lambda = \frac{p-q}{a_2p-a_1q} \quad , \quad x^2 = \frac{a_1-a_2}{a_1q-a_2p} \quad , \quad y = 0$$

and in the case of $(5.8)^-$ there is a curve passing through each of the two points

$$\Lambda = \frac{r-s}{a_2r-a_1s} \quad , \quad x = 0 \quad , \quad y^2 = \frac{a_1-a_2}{a_1s-a_2r} \quad .$$

Let the two curves of solutions of $(5.8)^+$ be denoted by $\Gamma_i^+ = \{(\Lambda_i^+(t),\ x_i^+(t),\ y_i^+(t)) : t \in (-1,1)\}$ where $y_i^+(0)=0$, $i = 1,2$ and let $\Gamma_i^- = \{(\Lambda_i^-(t),\ x_i^-(t),\ y_i^-(t)) : t \in (-1,1)\}$ where $x_i^-(0)=0$, $i = 1,2$ denote the solution curves of $(5.8)^-$. We can now apply the Implicit Function Theorem as in the previous case but this time using the space of continuous curves in $\mathbb{R}^3$ rather than the space of continuous closed curves. The result is that for $\sigma > 0$ sufficiently small there are two curves of solutions of $(5.8)^+$ close to Γ_i^+, $i = 1,2$ and two curves of solutions of $(5.8)^-$ close to Γ_i^-, $i = 1,2$. Clearly the former two curves each pass through a point at which $y = 0$ and the latter pass through a point at which $x = 0$. Then by rescaling we can recover solutions of (5.4) and from them obtain solutions of (N).

The result is that when $\beta > 0$ there are two secondary bifurcation points on C_β^N : one on each of ${}^{\pm}C_\beta^N$, and none on C_β^M. A calculation shows that each secondary bifurcation is a sub-critical pitchfork. Let $\mathcal{D}_\beta^1$ be the curve bifurcating from ${}^{+}C_\beta^N$ and $\mathcal{D}_\beta^2$ be that bifurcating from ${}^{-}C_\beta^N$. Clearly each $\mathcal{D}_\beta^i$ $(i = 1,2)$ is divided into two subsets by C_β^N : let

${}^{\pm}\mathcal{D}^i_\beta$ be the subsets of $\mathcal{D}^i_\beta$ (i = 1,2) corresponding to $\pm b > 0$. We shall see in Chapter VI that the symmetry observations of Chapter IV imply that the four sets ${}^{\pm}\mathcal{D}^i_\beta$ (i = 1,2) correspond to only two sets of capillary-gravity waves although they are all distinct solutions of $(\mathcal{N})$.

When $\beta < 0$ there are two sub-critical pitchfork bifurcations on C^M_β and none on C^N_β. As before, we label the secondary curves $\mathcal{D}^1_\beta$ and $\mathcal{D}^2_\beta$ according as they bifurcate from ${}^+C^M_\beta$ or ${}^-C^M_\beta$ respectively and ${}^{\pm}\mathcal{D}^i_\beta$ (i = 1,2) is the subset of $\mathcal{D}^i_\beta$ corresponding to $\pm a > 0$. As when $\beta > 0$ these four sets are linked to each other by symmetry. These results are pictured in Figure 2.

<u>5.3(b) $\beta = 0$</u>

In this case it follows from Theorem 5.1 that the curves C^M_0 and C^N_0 are still present and that they both bifurcate from $\alpha = 0$. (Note that even in this case each may still be regarded as the unique curve bifurcating from a simple eigenvalue if the problem is posed in X_M or X_N respectively.) However, since in the context of X_1, $\alpha = 0$ is a double eigenvalue there may be further primary curves and in any case there may be secondary bifurcation points on C^M_0 or C^N_0. In this case since $\beta = 0$, we can apply the Blowing-Up Lemma in a slightly different way than before in order to obtain a qualitative description of the solution set of (5.4) with $\beta = 0$ for (α,a,b) near $(0,0,0)$. Let m = 3, n = 2, $\underline{m} = (2,1,1)$, $\underline{n} = (2,2)$. The Taylor polynomial equation which results is

$$\begin{pmatrix} a_1\alpha + pa^2 + rb^2 \\ \\ a_2\alpha + qa^2 + sb^2 \end{pmatrix} = \underline{f}(\alpha,a,b) = 0 \tag{5.9}$$

and $\underline{Df}$ is given by

$$\begin{pmatrix} a_1, & 2ap, & 2br \\ \\ a_2, & 2aq, & 2bs \end{pmatrix} .$$

Clearly if $\underline{f}(\alpha,a,b) = 0$ and $(\alpha,a,b) \neq (0,0,0)$ then the rank of $D\underline{f}$ is two since at least one of a and b is non-zero and $a_1q-a_2p \neq 0$, $a_1s-a_2r \neq 0$. Hence by the Blowing-Up Lemma there is a smooth diffeomorphism between the solutions of (5.4) with $\beta = 0$ in a punctured neighbourhood of the origin and those of (5.9) in a punctured neighbourhood of the origin. It follows from (5.9) that any such solutions must satisfy

$$a^2 = \frac{(a_2r-a_1s)\alpha}{(ps-qr)} \quad , \quad b^2 = \frac{(a_1q-a_2p)\alpha}{(ps-qr)} \tag{5.10}$$

and by the formulae (4.41) - (4.43) it is clear that the situation is different according as $M > 2N$ or $M < 2N$. If $M > 2N$ it follows from these formulae that (5.9) has no solutions other than $(\alpha,a,b) = (0,0,0)$. Thus we conclude that when $\beta = 0$ and $M > 2N$ the bifurcation diagram of (N) in a neighbourhood of the origin consists solely of the primary curves C_0^M and C_0^N.

If $M < 2N$ then (5.10) represents two parabolae in $\mathbb{R}^3$ which intersect only once at the origin. Therefore, by the Blowing-Up Lemma the solution set of (5.4) in a neighbourhood of the origin must also consist of two curves intersecting at the origin. However to discover whether the bifurcation is super- or sub-critical we must again appeal to scaling techniques and the Implicit Function Theorem. This procedure is very similar to that used in the case $\beta \neq 0$. By similar arguments to those employed in the proof of Lemma 5.2 it follows that any small solutions of (5.4) with $ab = 0$ must satisfy the bounds $c_1'|\alpha| \le |(a,b)|^2 \le c_2'|\alpha|$ for positive constants c_1', c_2' and hence we introduce the scaling $(\alpha,a,b) = (\sigma^2\Lambda, \sigma x, \sigma y)$. After dividing by σ^2 equations (5.4) with $\beta = 0$ become

$$a_1\Lambda + px^2 + ry^2 + \bar{h}(\sigma, \Lambda, x, y) = 0 \tag{5.11a}$$

$$a_2\Lambda + qx^2 + sy^2 + \bar{k}(\sigma, \Lambda, x, y) = 0 \tag{5.11b}$$

where $\bar{h}$ and $\bar{k}$ are defined in the obvious way and enjoy the same properties as $h_\pm$ and $k_\pm$. When $\sigma = 0$ these equations are equivalent to

$$x^2 = \frac{(a_2r-a_1s)}{(ps-qr)}\Lambda \quad , \qquad y^2 = \frac{(a_1q-a_2p)}{(ps-qr)}\Lambda$$

which, as has just been observed, represent two parabolae passing through the origin. An examination of the signs of the coefficients yields that they occupy that part of $\mathbb{R}^3$ in which $\Lambda \le 0$. Let these curves be parameterised by

$$\Gamma_i = \{(\Lambda_i(t), x_i(t), y_i(t)) : t \in [-1,1]\} \qquad i = 1,2$$

where $\Lambda_i(0) = x_i(0) = y_i(0) = 0$. Then an application of the Implicit Function Theorem using the space of all curves in $\mathbb{R}^3$ yields that for $\sigma > 0$ sufficiently small there is a unique curve of solutions of (5.11) close to Γ_i, i = 1,2. These may be parameterised as $\{(\Lambda_i(\sigma,t),\ x_i(\sigma,t),\ y_i(\sigma,t)) : t \in [-1,1]\}$ and satisfy $\Lambda_i(\sigma,0) = x_i(\sigma,0) = y_i(\sigma,0) = 0$. Hence in the usual way we can recover solution curves of (5.4) and then of (N).

Therefore we conclude that when $\beta = 0$ and $M < 2N$ there are four primary curves of solutions of (N) all of which bifurcate from the origin: C_0^M , C_0^N and the two found here which we shall call $\mathcal{D}_0^1$ and $\mathcal{D}_0^2$ according as $ab > 0$ or $ab < 0$. Clearly each is divided into two subsets by the origin which we shall label ${}^{\pm}\mathcal{D}_0^i$ according as $\pm a > 0$. In Chapter VI it will be seen that the four sets ${}^{\pm}\mathcal{D}_0^i$, i = 1,2, correspond to only two distinct sets of capillary-gravity waves. It is already known that C_0^M and C_0^N are sub-critical bifurcations and the calculations performed here indicate that $\mathcal{D}_0^i$ (i = 1,2) are also. There are no known secondary bifurcations on any of these curves. See Figure 2.

This completes the analysis of the case $M \neq kN$.

There now follow some diagrams which illustrate the preceding analysis. The following key, which also applies to all subsequent diagrams, might facilitate their interpretation.

KEY

——— curve of solutions with corresponding $\theta \in X_M$

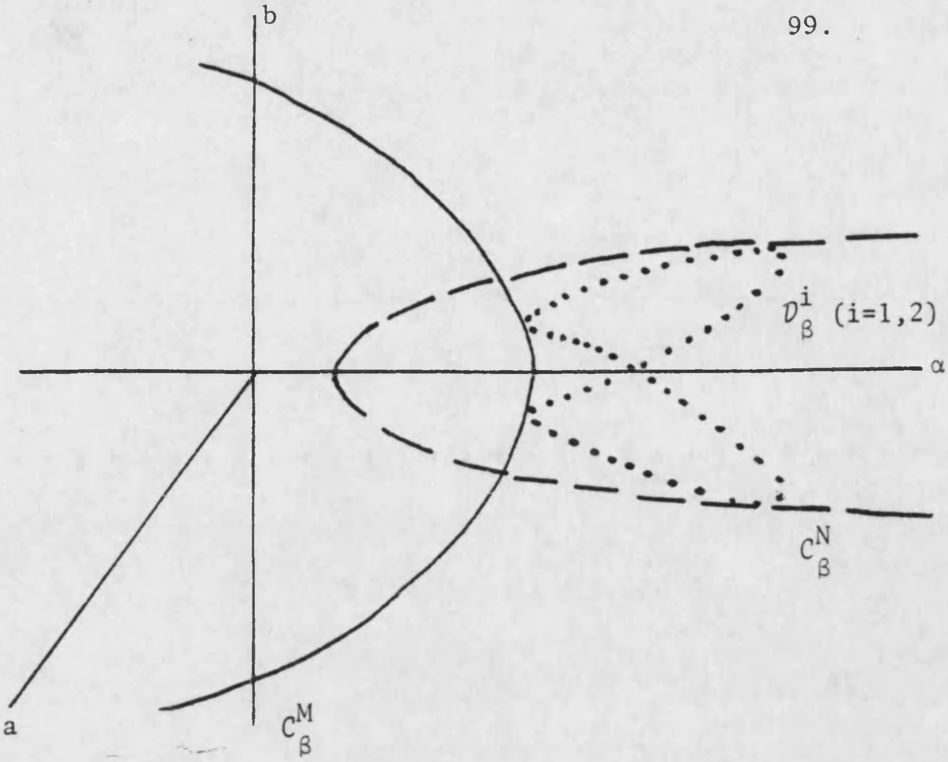
----- curve of solutions with corresponding $\theta \in X_N$

..... curve of solutions with corresponding $\theta \in X_K$.

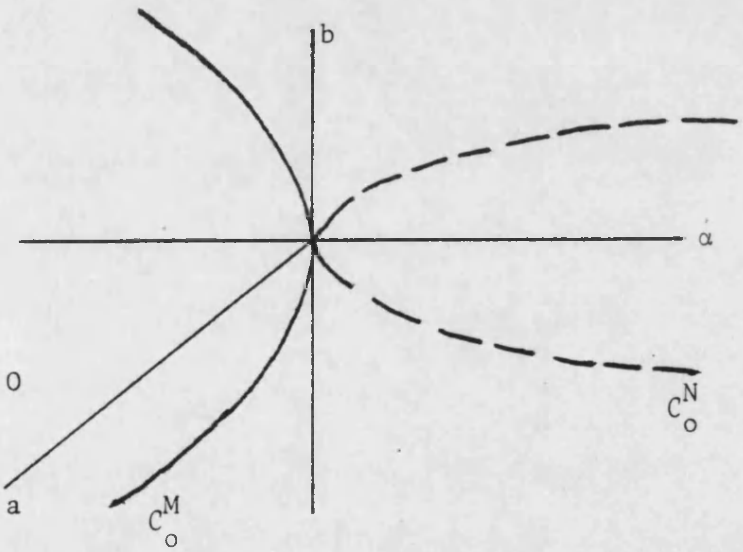
(Recall K is the highest common factor of M and N.)

Figure 1.

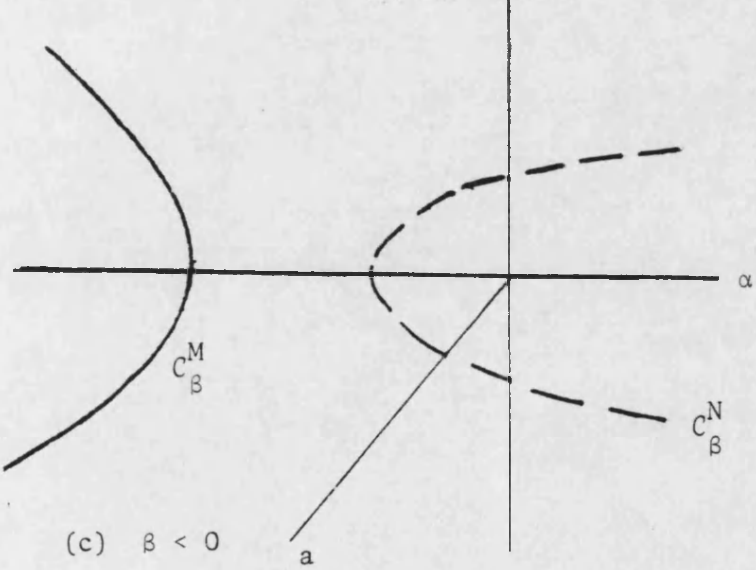
$M \neq kN, \quad M > 2N$



(a) $\beta > 0$



(b) $\beta = 0$



(c) $\beta < 0$

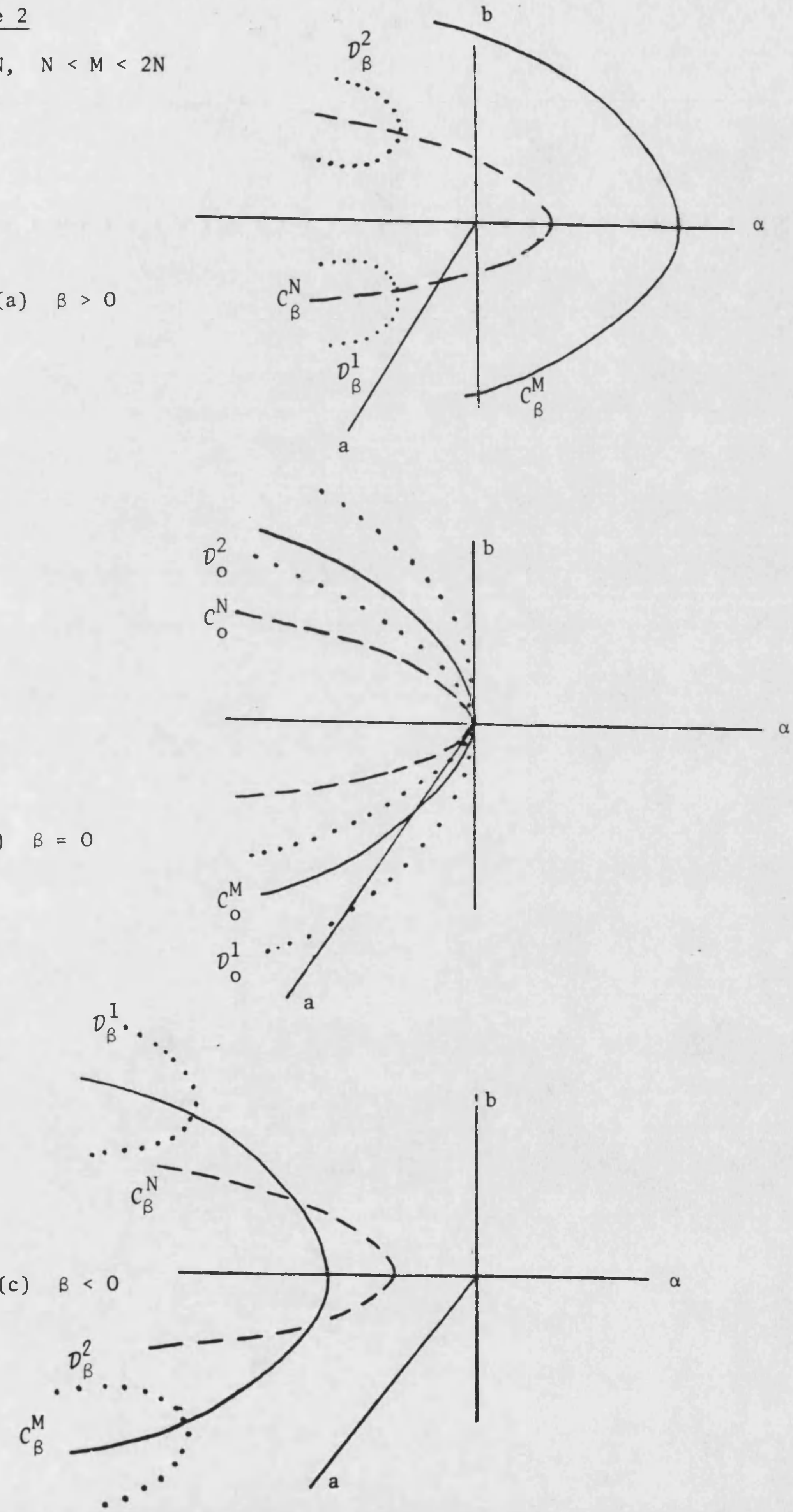
Figure 2

$M \neq kN, \quad N < M < 2N$

(a) $\beta > 0$

(b) $\beta = 0$

(c) $\beta < 0$



5.4 The Case M = kN

The bifurcation analysis in this case varies considerably according as $k = 2,3$ or ≥ 4 but we begin with a few general remarks common to all cases. The highest common factor of M and N is N and let us suppose throughout this section that k denotes the (fixed) positive number M/N (of course N could be 1 in which case M = k). Hence, since $X_M \subset X_N \subseteq X_1$ the problem (N) may be posed in the context of X_1 or X_N and the Lyapunov-Schmidt reduction is equally valid in each case. It follows then by uniqueness that all solutions θ found by this method lie in X_N (of course they may lie in smaller subspaces also) and correspond to capillary-gravity waves of period $2\pi/N$.

The basic difference between this and the previous case is that although the solution space of $\gamma_{NM}\theta = G_\theta(\nu_{NM},0)\theta$ is spanned by sin Ns and sin Ms there is no $n \in \mathbb{N}$ such that sin Ns $\in X_n$ but sin Ms $\notin X_n$. Certain of the symmetry arguments of §4.6 which led to the secondary bifurcation results of §5.3 fail in this case. The results (4.10) reduce to

$$-f_1(\alpha,\beta,a,b) = f_1(\alpha,\beta,-a,(-1)^k b),$$

$$(-1)^k f_2(\alpha,\beta,a,b) = f_2(\alpha,\beta,-a,(-1)^k b)$$

and $f_1(\alpha,\beta,0,b) = 0$ for all $(\alpha,\beta,0,b) \in U$.

Moreover, as was discussed in §4.6, there is no reason to believe that $f_2(\alpha,\beta,a,0)$ is zero for all $(\alpha,\beta,a,0) \in U$ and indeed it was shown in §4.9 that it is not zero when $k = 2,3,4,5$.

However since sin Ms $\in X_M$ and sin Ns $\notin X_M$, it is possible to consider (N) in the context of X_M. In this space, for fixed β, $\alpha = M\beta$

is always a simple eigenvalue with corresponding eigenfunction sin Ms. We then have the following version of Theorem 5.1.

THEOREM 5.3. *Suppose* $M = kN$ *for some* $k \in \mathbb{N}$. *Then there exists* $d > 0$ *and a continuous function* $\alpha_M : R_d \to \mathbb{R}$ *which satisfies* $\alpha_M(\beta,0) = M\beta$ *and is such that for fixed* $\beta = (-d,d)$ *the curve*

$$C^M_\beta = \{(\alpha_M(\beta,b),\ b \sin Ms + z(\alpha_M,\beta,0,b)) : |b| < d\}$$

is a primary curve of solution of (N) *bifurcating from* $\alpha = M\beta$.

Moreover there is a $\delta > 0$ *such that if* $(\alpha,\beta,\theta) \in \mathbb{R}\times\mathbb{R}\times X_M$ *is a solution of* (N) *satisfying* $|\alpha| < \delta$, $|\beta| < d$, *and* $||\theta|| < \delta$ *then either* $\theta \equiv 0$ *or* $(\alpha,\theta) \in C^M_\beta$.

Remarks

1. A calculation yields that C^M_β is always given by (5.3) whatever the relationship between M and N. Therefore in all cases it is a subcritical pitchfork. As before we shall denote the branches of the pitchfork by ${}^{\pm}C^M_\beta$ according as $\pm b > 0$ and we shall see in Chapter VI that both branches correspond to the same set of capillary-gravity waves.

2. When $\beta \neq 0$, $\alpha = N\beta$ is a simple eigenvalue with eigenspace spanned by sin Ns. Therefore by Theorem 4.1 there is a unique branch of solutions bifurcating from it which we shall denote by $\tilde{C}^N_\beta$. However unlike the corresponding branch C^N_β when $M \neq kN$ this does not correspond to solutions of the bifurcation equations which have $b = 0$. Further although $\tilde{C}^N_\beta \subset \mathbb{R}\times X_N$ (since in this section any solution $\theta \in X_N$), there is no $n > N$ such that $\tilde{C}^N_\beta \subset \mathbb{R}\times X_n$. This is in contrast to the situation

when $M \neq kN$ where $C_\beta^N \subset \mathbb{R}\times X_N \lhd \mathbb{R}\times X_K$, (K being the highest common factor of M and N). We now wish to determine the structure of $\tilde{C}_\beta^N$ and also the existence of any secondary bifurcation points. Since all solutions of the bifurcation equations with a = 0 have been accounted for (corresponding as they do to solutions of (N) which either belong to C_β^M or are trivial); it suffices to study equations $(\mathcal{B}1)$ divided by a and $(\mathcal{B}2)$. These equations are (see §4.8)

$$\beta = a_1\alpha + b_1 b + pa^2 + tab + rb^2 + h_1(\alpha,\beta,a,b),$$
$$\beta b = a_2\alpha b + b_2 a^2 + ua^3 + qa^2 b + sb^3 + k(\alpha,\beta,a,b).$$

We now study the various cases separately.

5.4(a) Bifurcation when M/N ≥ 4

In this case the formulae of §4.8 dictate that the equations to be studied are

$$\beta = a_1\alpha + pa^2 + rb^2 + h_1(\alpha,\beta,\ a,\ b), \tag{5.12a}$$

$$\beta b = a_2\alpha b + qa^2 b + sb^3 + k(\alpha,\beta,\ a,b), \tag{5.12b}$$

where a_1, a_2, etc. have precisely the same values in terms of M and N as they did in §5.3. Indeed the only difference between those equations and the ones studied here is that

$$k(\alpha,\beta,\ a,\ 0) \neq 0 \qquad \text{for all} \quad (\alpha,\beta,\ a,\ 0) \in U.$$

(To be quite precise this is true (see §4.9) when k = 4 and 5 and it would seem likely to be true in general.)

Thus the analysis of these equations is similar, but not the same, as the analysis of the bifurcation equations when $M \neq kN$, $M > 2N$. First observe that by an almost identical application of the Blowing-Up Lemma to that appearing in §5.3, but this time with $m = 4$, $n = 2$, $\underline{m} = (2,2,1,1)$, $\underline{n} = (2,3)$ we can deduce that it is sufficient to study equations (5.12) truncated at the cubic level. As before we consider the different values of β separately.

<u>The Case $\beta \neq 0$</u>

As explained in the second of the remarks following Theorem 5.3, we know that in this case there is a primary curve, C_β^{2N} bifurcating from $\alpha = N\beta$. We wish to determine the approximate form of this curve and also to see whether there are any secondary bifurcation points on C_β^M.

To do this the first step is to find all solutions of (5.12) which have $ab = 0$. An argument almost identical to that used in Lemma 5.2 shows that any such solutions must satisfy the same "a-priori" bounds and hence the same scaling may be used.

The resulting equations are therefore

$$\pm 1 = a_1\Lambda + px^2 + ry^2 + h_\pm(\sigma,\Lambda,x,y), \qquad (5.13a)^\pm$$

$$\pm y = a_2\Lambda y + qx^2y + sy^3 + \overline{k}_\pm(\sigma,\Lambda,x,y), \qquad (5.13b)^\pm$$

where $\overline{k}_\pm(\sigma,\Lambda,x,y) = \sigma^{-3}k(\sigma^2\Lambda, \pm\sigma^2, \sigma x, \sigma y)$ and is defined as zero for $\sigma = 0$. As before the procedure is to find solutions of $(5.13)^\pm$ with $\sigma = 0$ which pass through $xy = 0$ and then use the Implicit Function

Theorem to deduce the existence of such solutions for $\sigma > 0$ sufficiently small. When $\sigma = 0$, $(5.13)^{\pm}$ always has the solution

$$\left\{(\hat{\Lambda},\hat{x},\hat{y})\right\} = \left\{ \left(-\frac{1}{a_1}(\pm 1 + px^2),\ x,\ 0 \right) : x \in \mathbb{R} \right\}.$$

Now let C be the Banach space of continuous curves in $\mathbb{R}^3$ and define $G : \mathbb{R}\times C \to C$ by

$$G(\sigma,(\Lambda,x,y)) = \begin{pmatrix} a_1\Lambda + px^2 + ry^2 + h_{\pm}(\sigma,\Lambda,x,y) \\ \\ a_2\Lambda y \mp y + qx^2y + sy^3 + \overline{k}_{\pm}(\sigma,\Lambda,x,y) \\ \\ x \end{pmatrix}.$$

Then the determinant of the derivative of G with respect to (Λ,x,y) evaluated at $(\sigma,\Lambda,x,y) = (0,\hat{\Lambda},\hat{x},\hat{y})$ is $a_1(a_2\hat{\Lambda} \pm 1 + q\hat{x}^2)$ which is non-zero for all values of x sufficiently close to zero. Hence by the Implicit Function Theorem, for $\sigma > 0$ sufficiently small there is a unique solution curve of

$$G(\sigma,(\Lambda,x,y)) = \begin{pmatrix} \pm 1 \\ 0 \\ \hat{x} \end{pmatrix}$$

passing through the point $(\pm\frac{1}{a_1}, 0, 0)$. Hence by the usual methods this corresponds to the primary bifurcation branch $\tilde{C}^N_\beta$ which is revealed to bifurcate super-critically. As before we shall denote the two subsets of $\tilde{C}^N_\beta \setminus \{(N\beta,0)\}$ by ${}^{\pm}\tilde{C}^N_\beta$ according as $\pm a > 0$, and we shall see in Chapter VI that these two solution branches correspond to the same set of capillary-gravity waves. When $\sigma = 0$ (5.13)$^-$ has no other solutions. Hence there is no secondary bifurcation on C_β^M when $\beta < 0$.

However, when $\sigma = 0$, (5.13)$^+$ has a closed loop of solutions (indeed the equations are the same as those treated in §5.3 when $\sigma = 0$ once the second has been divided through by y). Let this loop be parameterised by $\{(\hat{\Lambda}(t), \hat{x}(t), \hat{y}(t)) : t \in [0,2\pi]\}$ and suppose it passes through the points at which $x = 0$ at $t = \pi/2$, $3\pi/2$ and the points at which $y = 0$ at $t = 0, \pi$. Further, set

$$\Gamma_1 = \{(\hat{\Lambda}(t), \hat{x}(t), \hat{y}(t)) : t \in (0,\pi)\}$$

$$\Gamma_2 = \{(\hat{\Lambda}(t), \hat{x}(t), \hat{y}(t)) : t \in (\pi,2\pi)\}.$$

Now define $F : \mathbb{R}\times C \to C$ by

$$F(\sigma,(\Lambda,x,y)) = \begin{pmatrix} a_1\Lambda + px^2 + ry^2 + h_+(\sigma,\Lambda,x,y) \\ a_2\Lambda y - y + qx^2y + sy^3 + \bar{k}_+(\sigma,\Lambda,x,y) \\ 2(ps-qr)\hat{x}\hat{y}\Lambda + (a_2r-a_1s)x\hat{y} + (a_1q-a_2p)\hat{x}y \end{pmatrix} .$$

Then a calculation shows that the determinant of the derivative of F with respect to (Λ,x,y) evaluated at $(0,\hat{\Lambda},\hat{x},\hat{y})$ is $2\hat{y}(\hat{x}^2(a_1q-a_2p)^2 + \hat{y}^2(a_1s-a_2r)^2 + 4\hat{x}^2\hat{y}^2(ps-qr)^2)$ which is non-zero except at the points at which $\hat{y} = 0$.

Hence the Implicit Function Theorem yields that for $\sigma > 0$ sufficiently small there are unique solutions of

$$F(\sigma,(\Lambda,x,y)) = \begin{pmatrix} 1 \\ \\ 0 \\ \\ \hat{x}\hat{y}(2(ps-qr)\hat{\Lambda}+(a_2r-a_1s) + (a_1q-a_2p)) \end{pmatrix}$$

close to Γ_1 and Γ_2. Let these be denoted by Γ_1^* and Γ_2^*. Then it is clear that by an argument similar to that used in §5.3 that each x-variable in Γ_1^* and Γ_2^* is zero exactly once. Hence by rescaling we can recover two solution curves of (5.12) in $\mathbb{R}^3$, which lie on opposite sides of the plane b = 0 and along each of them a = 0 exactly once. Each curve comes arbitrarily close to the b = 0 plane but in general we do not believe it cuts it. Indeed since when k = 4 and 5 it is known that (5.12) has no small solutions with b = 0 this is definitely true in these special cases and it seems likely to be true in the general case also.

Then from these curves of solutions in $\mathbb{R}^3$ we can obtain solution sets of $(\alpha,\theta) \in \mathbb{R}\times X_N$ of (N). The points at which a = 0 correspond to secondary bifurcation points on C_β^M and, as before, both bifurcations are super-critical. It is not known whether the secondary curve intersects the other primary curve, $\tilde{C}_\beta^N$ (recall that this curve does not, in this case, correspond to solutions of (5.12) for which b = 0) but in general this seems unlikely. As before we denote by $\mathcal{D}_\beta^1$ the secondary curve which bifurcates from ${}^+C_\beta^M$ and by $\mathcal{D}_\beta^2$ that which bifurcates from ${}^-C_\beta^M$. Each secondary curve is divided into two subsets by the primary curve and we label these subsets ${}^\pm\mathcal{D}_\beta^i$ according as $\pm a > 0$ along them. See Figure 3.

(Later in Chapter VI we shall see that these four sets correspond to only two distinct families of capillary-gravity waves.)

The case $\beta = 0$

In an analogous way to the case when $M \neq kN$ we can use the Blowing-Up Lemma when $m = 4$, $n = 2$, $\underline{m} = (2,2,1,1)$ and $\underline{n} = (2,3)$ to deduce that there is a smooth diffeomorphism between the solution set of (5.12) with $\beta = 0$ in a punctured neighbourhood of the origin in $\mathbb{R}^3$ and that of

$$a_1\alpha + pa^2 + rb^2 = 0 \tag{5.14a}$$

$$a_2\alpha b + qa^2b + sb^3 = 0 \tag{5.14b}$$

in another punctured neighbourhood of the origin. The only solution of (5.14) is

$$\left\{(\alpha,\ a,\ b)\right\} = \left\{ \left(-\frac{pa^2}{a_1},\ a,\ 0 \right) \ :\ a \in \mathbb{R} \right\}$$

and hence there is one solution curve of (5.12) which passes through the origin. To determine the approximate form of this curve (i.e. to see if the bifurcation is super- or sub-critical) we should use scaling techniques and the Implicit Function Theorem as before. It turns out that the solution curve is given by $\tilde{C}_0^N$ i.e. $\tilde{C}_\beta^N$ with $\beta = 0$.

Thus in this case the solution set of (N) in a neighbourhood of the origin consists of two primary curves bifurcating from $\alpha = 0$: $C_0^M \subset \mathbb{R}\times X_M$ which bifurcates sub-critically and $\tilde{C}_0^N \subset \mathbb{R}\times X_N$ which bifurcates super-critically. See Figure 3.

5.4(b) Bifurcation when $M = 3N$

As usual there is a primary solution branch $C_\beta^{3N} \subset \mathbb{R}\times X_{3N}$ which bifurcates sub-critically from $\alpha = 3N\beta$ for all values of β and it is given by (5.3). To determine any other solutions the equations to be studied are

$$\beta = a_1\alpha + pa^2 + tab + rb^2 + h_1(\alpha,\beta,a,b) \tag{5.15a}$$

$$\beta b = a_2\alpha b + ua^3 + qa^2 b + sb^3 + k(\alpha,\beta,a,b). \tag{5.15b}$$

(Where a_1, a_2, etc. are given in terms of N in §4.8.)
Note that (5.15b) contains a term in a^3 and thus has no non-zero small solutions for which $b = 0$. It is a routine matter to show that the Blowing-Up Lemma holds when $m = 4$, $n = 2$, $\underline{m} = (2,2,1,1)$ and $\underline{n} = (2,3)$ and thus it is sufficient to study the truncated equations. As usual the cases $\beta = 0$ and $\beta \neq 0$ will be considered separately.

<u>The Case $\beta \neq 0$</u>

We already know that since $\alpha = N\beta$ is a simple eigenvalue there must be a primary curve of solutions of (N) bifurcating from this point. In addition we wish to investigate the possibility of secondary bifurcation occurring on C_β^{3N}. Such points correspond to solutions of (5.15) which have $ab = 0$ and an elementary calculation shows that the same " a priori" bounds and scaling as before can be used. The resulting system is (on substituting numerical values for the coefficients)

$$\pm 1 = \frac{\Lambda}{N} - \frac{77}{24N^2}x^2 - \frac{41}{8N^2}xy - \frac{23}{36N^2}y^2 + h_\pm(\sigma,\Lambda,x,y) \tag*{(5.16a)$^\pm$}$$

$$0 = \frac{\Lambda}{3N}y \mp y - \frac{41}{24N^2}x^3 - \frac{23}{36N^2}x^2y + \frac{29}{360N^2}y^3 + \overline{k}_\pm(\sigma,\Lambda,x,y). \tag*{(5.16b)$^\pm$}$$

As before the first step is to find solutions of (5.16) when $\sigma = 0$. These equations always have the solutions $\Lambda = \pm N$, $x = y = 0$. We can now use the Implicit Function Theorem in the following way to show that this solution corresponds to the primary bifurcation from $\alpha = N\beta$ and also to determine the nature of this bifurcation.

Define the map $G^{\pm} : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$G^{\pm}(\sigma,\Lambda,x,y) = \begin{pmatrix} \frac{\Lambda}{N} - \frac{77}{24N^2}x^2 - \frac{41}{8N^2}xy - \frac{23}{36N^2}y^2 + h_{\pm}(\sigma,\Lambda,x,y) \\ \\ \frac{\Lambda}{3N}y \mp y - \frac{41}{24N^2}x^3 - \frac{23}{36N^2}x^2y + \frac{29}{360N^2}y^3 + \overline{k}_{\pm}(\sigma,\Lambda,x,y) \end{pmatrix}.$$

Then $G^{\pm}(0,\pm N,0,0) = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$ and the determinant of $\dfrac{\partial G^{\pm}}{\partial(\Lambda,y)}(0, \pm N,0,0)$

is $\pm\frac{2}{3N}$ which is non-zero. Hence there is a $\delta > 0$ and functions

$\Lambda^{\pm}(\sigma,x)$, $y^{\pm}(\sigma,x)$ each from $(-\delta,\delta)\times(-\delta,\delta)$ to $\mathbb{R}$ which satisfy $\Lambda^{\pm}(0,0) = \pm N$ and $y^{\pm}(0,0) = 0$ and are also such that

$$G(\sigma,\Lambda^{\pm}(\sigma,x),x,y^{\pm}(\sigma,x)) = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \quad \text{for } (\sigma,x) \in (-\delta,\delta)\times(-\delta,\delta). \qquad (5.17)$$

These solutions clearly correspond, via the usual rescaling to the primary curve $\tilde{C}^N_\beta$ which bifurcates from $\alpha = N\beta$ and whose existence is already known. As usual we shall denote the subsets of $\tilde{C}^N_\beta$ by $^{\pm}\tilde{C}^N_\beta$ according as $\pm a > 0$. Further, on differentiating (5.17) twice and setting $\nu = x = 0$ we obtain $\Lambda^{\pm}_{xx}(0,0) = \frac{77}{12N}$ which indicates that $\tilde{C}^N_\beta$ is a super-critical bifurcation. Any secondary bifurcations on C^{3N}_β correspond to solutions of (5.16) with $\sigma = 0$ which have $x = 0$, $y \neq 0$. $(5.16)^-$ has no such solutions and consequently when $\beta < 0$ there are no secondary bifurcation points on C^{3N}_β.

However $(5.16)^+$ has the solutions $\Lambda_o = (\frac{777}{317})N$, $y_o^2 = (\frac{720}{317})N^2$, $x = 0$ and these are the only solutions for which $x = 0$. Then, as before the Implicit Function Theorem can be used to show the existence of $\delta > 0$ and functions Λ_i, y_i : $(-\delta,\delta)\times(-\delta,\delta) \to \mathbb{R}$ which satisfy $\Lambda_i(0,0) = \Lambda_o$ and $y_i(0,0) = (-1)^i y_o$, $i = 1,2$ and are such that (σ,Λ_i,x,y_i) is the unique solution of $G^+(\sigma, \Lambda, x,y) = (\frac{1}{0})$ in a neighbourhood of $(0, \Lambda_o, 0, (-1)^i y_o)$. Then, by rescaling, these curves correspond to two solutions curves of (5.15) in $\mathbb{R}^3$ along both of which $a = 0$ exactly once and hence to two secondary bifurcation curves of (N) : one, $\mathcal{D}_\beta^1$, which bifurcates from ${}^+C_\beta^{3N}$ and the other $\mathcal{D}_\beta^2$ from ${}^-C_\beta^{3N}$. As usual we shall label by ${}^\pm\mathcal{D}_\beta^i$ the subsets of $\mathcal{D}_\beta^i$ along which $\pm a > 0$. (We shall see in Chapter VI that these four subsets correspond to only two distinct sets of capillary-gravity waves.) A further calculation yields that $\frac{d}{dx}\Lambda_i(0,0) = (-1)^i \frac{3567}{2536} y_o$ and hence the secondary bifurcation is transcritical.

To summarise: if $M = 3N$ and $\beta \neq 0$ there are two primary bifurcation curves. One, $\mathcal{C}_\beta^N$ bifurcates from $\alpha = N\beta$, while the other $C_\beta^{3N} \subset \mathbb{R}\times X_{3N}$ and bifurcates super-critically from $\alpha = 3N\beta$. If $\beta < 0$, these are the only solution curves present. However if $\beta > 0$ then there are two transcritical secondary bifurcations: one from each of ${}^\pm C_\beta^{3N}$. See Figure 4.

The Case $\beta = 0$

The equations to be studied are now

$$a_1\alpha + pa^2 + tab + rb^2 + h_1(\alpha,\beta,a,b) = 0 \tag{5.18a}$$

$$a_2\alpha b + ua^3 + qa^2b + sb^3 + k(\alpha,\beta,a,b) = 0. \tag{5.18b}$$

As usual the first step is to apply the Blowing-Up Lemma. However, this is a more complicated procedure than hitherto. Let $m = 3$, $n = 2$, $\underline{m} = (2,1,1)$, $\underline{n} = (2,3)$. The resulting Taylor polynomial equation is

$$\begin{pmatrix} a_1\alpha + pa^2 + tab + rb^2 \\ a_2\alpha b + ua^3 + qa^2b + sb^3 \end{pmatrix} = \underline{f}(\alpha,a,b) = 0. \tag{5.19}$$

Now $D\underline{f}$ is

$$\begin{pmatrix} a_1, & 2pa+tb, & ta+2rb \\ a_2b, & 3ua^2+2qab, & a_2\alpha+qa^2+3sb^2 \end{pmatrix} .$$

If $a = 0$ and $b \neq 0$ and $\underline{f}(\alpha,a,b) = 0$ then $D\underline{f}$ has rank two since the first two columns are linearly independent. There are no solutions of $\underline{f} = 0$ with $a \neq 0$, $b = 0$. If $\underline{f}(\alpha,a,b) = 0$ and $ab \neq 0$, $D\underline{f}$ reduces to

$$\begin{pmatrix} a_1, & 2pa+tb, & ta+2rb \\ a_2b, & 3ua^2+2qab, & 2sb^2-ua^3b^{-1} \end{pmatrix} .$$

A necessary condition for the rank to be less than two is that the subdeterminant of the first and second and first and third columns be simultaneously zero. This amounts to the following equations for $v = (b/a)$ having coincident solutions:

$$a_2tv^2 + 2(a_2p - a_1q)v - 3a_1u = 0,$$

$$2(a_1s - a_2r)v^3 - a_2tv^2 - a_1u = 0.$$

Substituting numerical values for the coefficients yields the equations:

$$\frac{41}{2}v^2 + \frac{31}{3}v - \frac{123}{2} = 0,$$

$$\frac{143}{45}v^3 + \frac{41}{2}v^2 + \frac{41}{2} = 0.$$

The solutions to the first equation are approximately -2.0,-1.5 and these do not satisfy the second. Hence the hypotheses of the Blowing-Up Lemma are satisfied. However, it is still necessary to use scaling arguments and the Implicit Function Theorem to obtain the approximate form of the solution set of (5.18) in a neighbourhood of the origin. Any solutions of these equations with ab = 0 must satisfy the same "a priori" bounds as in previous sections and consequently we are able to use the same scaling arguments. The equations become

$$a_1\Lambda + px^2 + txy + ry^2 + \tilde{h}(\sigma,\Lambda,x,y) = 0 \tag{5.20a}$$

$$a_2\Lambda y + ux^3 + qx^2y + sy^3 + \tilde{k}(\sigma,\Lambda,x,y) = 0 \; . \tag{5.20b}$$

First note that there are no solutions of these equations with $\sigma = 0$, for which x = 0, and Λ or y are non-zero. Then putting $\sigma = 0$, eliminating Λ between (5.20a) and (5.20b) (and substituting numerical values for the coefficients) there results the following equation for $w = (y/x)$:

$$\frac{317}{45}w^3 + 41w^2 + \frac{31}{3}w - 41 = 0 \ .$$

A calculation yields that this equation has three real roots

$$w_1 \sim -5.3, \quad w_2 \sim -1.3, \quad w_3 \sim 0.83$$

and to each root there corresponds a branch

$$\Gamma_i = \left\{(\Lambda_i, x_i, y_i)\right\} = \left\{\left(-\frac{x^2}{a_1}(p + tw_i + rw_i^2),\ x, xw_i\right) : x \in \mathbb{R}\right\} \ i = 1,2,3 \tag{5.21}$$

of solutions of (5.20) when $\sigma = 0$.

We can obtain the three corresponding solution curves of (5.20) when $\sigma > 0$ by the following method. Let C be the Banach space of curves in $\mathbb{R}^3$ and let $F^i(\sigma,(\Lambda,x,y)) : \mathbb{R}\times C \to C$ be defined by

$$F^i(\sigma,(\Lambda,x,y)) = \begin{pmatrix} a_1\Lambda + px^2 + txy + ry^2 + \tilde{h}(\sigma,\Lambda,x,y) \\ a_2\Lambda y + ux^3 + qx^2y + sy^3 + \tilde{k}(\sigma,\Lambda,x,y) \\ (a_1q - a_2p)x_iy \end{pmatrix} .$$

Then

$$F^i(0,(\Lambda_i,x_i,y_i)) = \begin{pmatrix} 0 \\ 0 \\ (a_1q - a_2p)x_iy_i \end{pmatrix}$$

and the derivative of $F^i(\sigma,(\Lambda,x,y))$ with respect to (Λ,x,y) evaluated at $(\sigma,\Lambda,x,y) = (0,\Lambda_i,x_i,y_i)$ is

$$2(a_1q-a_2p)^2x_i^2y_i + 3a_1ux_i^3 - a_2tx_iy_i^2$$

$$= x_i^3(2(a_1q-a_2p)^2w_i + 3a_1u - a_2tw_i^2) \qquad \text{(since } y_i = w_ix_i,\ i = 1,2,3)$$

$$\sim 111x_1^3,\ -11x_2^3,\ -9x_3^3 \qquad \text{(according as } i = 1,2,3)$$

which are all clearly non-zero except when x_i (and hence Λ_i and y_i also) is zero.

Hence for $\sigma > 0$ sufficiently small there is a unique curve of solutions of (5.20) close to Γ_i, which we shall denote by Γ_i^σ $(i = 1,2,3)$. Since $(\Lambda,x,y) = (0,0,0)$ is a solution of (5.20) it follows that each of these curves passes through the origin. Hence by rescaling we obtain three solution curves of (5.18) and hence three primary solution curves of (N) all of which bifurcate from $\alpha = 0$ and which we shall label C_0', C_0'' and C_0'''. An elementary calculation based on (5.21) shows that two of these bifurcate sub-critically while the remaining one bifurcates super-critically. Clearly each curve is divided into two subsets by the origin and we shall see in Chapter VI that both subsets correspond to the same set of capillary-gravity waves.

To summarise then: when $\beta = 0$ and $M = 3N$ there is primary bifurcation of four solution curves from the origin. Three of these bifurcate sub-critically, while the remaining one bifurcates super-critically. One of the sub-critical branches, C_0^{3N} is a subset of $\mathbb{R}\times X_{3N}$, while the other bifurcating curves are subsets of $\mathbb{R}\times X_N$ but not of $\mathbb{R}\times X_n$ for any $n > N$. There are no secondary bifurcations on any of these curves. See Figure 4.

5.4(c) Bifurcation when M = 2N

For all values of β, C_β^{2N} bifurcates sub-critically from $\alpha = 2N\beta$. An inspection of the formulae in §4.8 yields that to find any other primary or secondary bifurcations we must consider the equations

$$\beta = \frac{\alpha}{N} - \frac{3b}{4N^2} + \frac{3a^2}{16N^2} + \frac{21b^2}{16N^2} + h_1(\alpha,\beta,a,b) \tag{5.22a}$$

$$\beta b = \frac{\alpha b}{2N} - \frac{3a^2}{8N^2} + \frac{3b^3}{16N^2} + \frac{9a^2 b}{8N^2} + k(\alpha,\beta,a,b) \quad . \tag{5.22b}$$

Note that unlike the previous cases, the bifurcation equations now contain terms quadratic in a and b. It is a straightforward calculation to verify that the hypotheses of the Blowing-Up Lemma are satisfied when $m = 4$, $n = 2$, $\underline{m} = (2,2,1,1)$ and $\underline{n} = (2,3)$ and so there is a smooth diffeomorphism between the solutions of (5.22) in a punctured neighbourhood of the origin in $\mathbb{R}^4$ and those of the truncated equations in another punctured neighbourhood of the origin. As usual the different values of β are given separate consideration.

The Case $\beta \neq 0$

We know the existence of $\tilde{C}_\beta^N$ which bifurcates from $\alpha = N\beta$. We wish to determine its approximate form and also to see if there is any secondary bifurcation from C_β^{2N}. To do this it is necessary to find solutions of (5.22) for which $ab = 0$ and as usual this can be done using scaling techniques and the Implicit Function Theorem. However an argument similar to that used in Lemma 5.2 indicates that the appropriate scaling to use is $(\alpha,\beta,a,b) = (\sigma\Lambda, \pm\sigma, \sigma x, \sigma y)$, where Λ,x,y are bounded and σ is small.

Equations (5.22) then become

$$\pm 1 = \frac{\Lambda}{N} - \frac{3y}{4N^2} + \underline{h}_{\pm}(\sigma,\Lambda,x,y) \qquad (5.23a)^{\pm}$$

$$\pm y = \frac{\Lambda y}{2N} - \frac{3x^2}{8N^2} + \underline{k}_{\pm}(\sigma,\Lambda,x,y) \qquad (5.23b)^{\pm}$$

where $\underline{h}_{\pm}$ and $\underline{k}_{\pm}$ are defined in an obvious way.

When $\sigma = 0$ each of $(5.23)^{\pm}$ has two solution curves. $(5.23)^{+}$ has

$$\left\{ x^2 = \frac{16N^2}{9}(\rho-2N)(\rho-N),\ y = \frac{4N}{3}(\rho-N),\ \rho \geq 2N \text{ and } \rho \leq N \right\} \qquad (5.24)^{+}$$

which represents two curves, one of which passes through the point

$$\rho = N, \quad x = y = 0$$

and the other through the point

$$\rho = 2N,\ x = 0,\ y = \frac{4N^2}{3} \ .$$

$(5.23)^{-}$ has the solution curves

$$\left\{ x^2 = \frac{16N^2}{9}(\rho+2N)(\rho+N),\ y = \frac{4N}{3}(\rho+N),\ \rho \geq -N \text{ and } \rho \leq -2N \right\} \qquad (5.24)^{-}$$

which pass through

$$\rho = -N, \quad x = y = 0$$

and

$$\rho = -2N,\ x = 0,\ y = -\frac{4}{3}N^2 \ ,$$

respectively.

Then the Implicit Function Theorem applied in a now familiar way shows that in each case the curve passing through $x = y = 0$ corresponds to the primary curve $\tilde{C}^N_\beta$. Furthermore, it is clear from $(5.24)^+$ that when $\beta > 0$ this is a sub-critical pitchfork while when $\beta < 0$ it is a super-critical pitchfork. In both cases the other solution curve corresponds to a single secondary bifurcation point on C^{2N}_β and the formulae (5.24) yield that when $\beta > 0$ this is a super-critical pitchfork from ${}^+C^{2N}_\beta$ while when $\beta < 0$ it is a sub-critical pitchfork from ${}^-C^{2N}_\beta$. See Figure 5.

<u>The Case $\beta = 0$</u>

A straightforward application of the Blowing-Up Lemma when $m = 4$, $n = 2$, $\underline{m} = (2,2,1,1)$, $\underline{n} = (1,2)$ yields that in a neighbourhood of the origin in $\mathbb{R}^3$ the solution set of (5.22) with $\beta = 0$ is qualitatively the same as that of

$$\frac{\alpha}{N} - \frac{3b}{4N^2} = 0$$

$$\frac{\alpha b}{2N} - \frac{3a^2}{8N^2} = 0.$$

These equations have the solution curves

$$b = (\tfrac{4}{3})N\alpha, \quad a^2 = (\tfrac{16}{9})N^2\alpha^2 \tag{5.25}$$

which represent two straight lines through the origin and a now familiar application of scaling techniques and the Implicit Function Theorem yields that these are indeed close to the solutions of (5.22) with $\beta = 0$. Hence when $\beta = 0$ the bifurcation diagram of (N) consists of three primary curves all of which bifurcate at $\alpha = 0$. One, $C^{2N}_0 \subset \mathbb{R}\times X_{2N}$, is a sub-critical pitchfork. The other two, which we shall label C'_0 and C''_0, correspond to (5.25). $C'_0 \cup C''_0$ consist of two transcritical bifurcations from $\alpha = 0$. However as is suggested by the analysis when $\beta \neq 0$ and as we shall see later from symmetry considerations these are better regarded as two degenerate parabolae, one super- and one sub-critical bifurcating from $\alpha = 0$. See Figure 5.

Figure 3

$M = kN$, $k \geq 4$

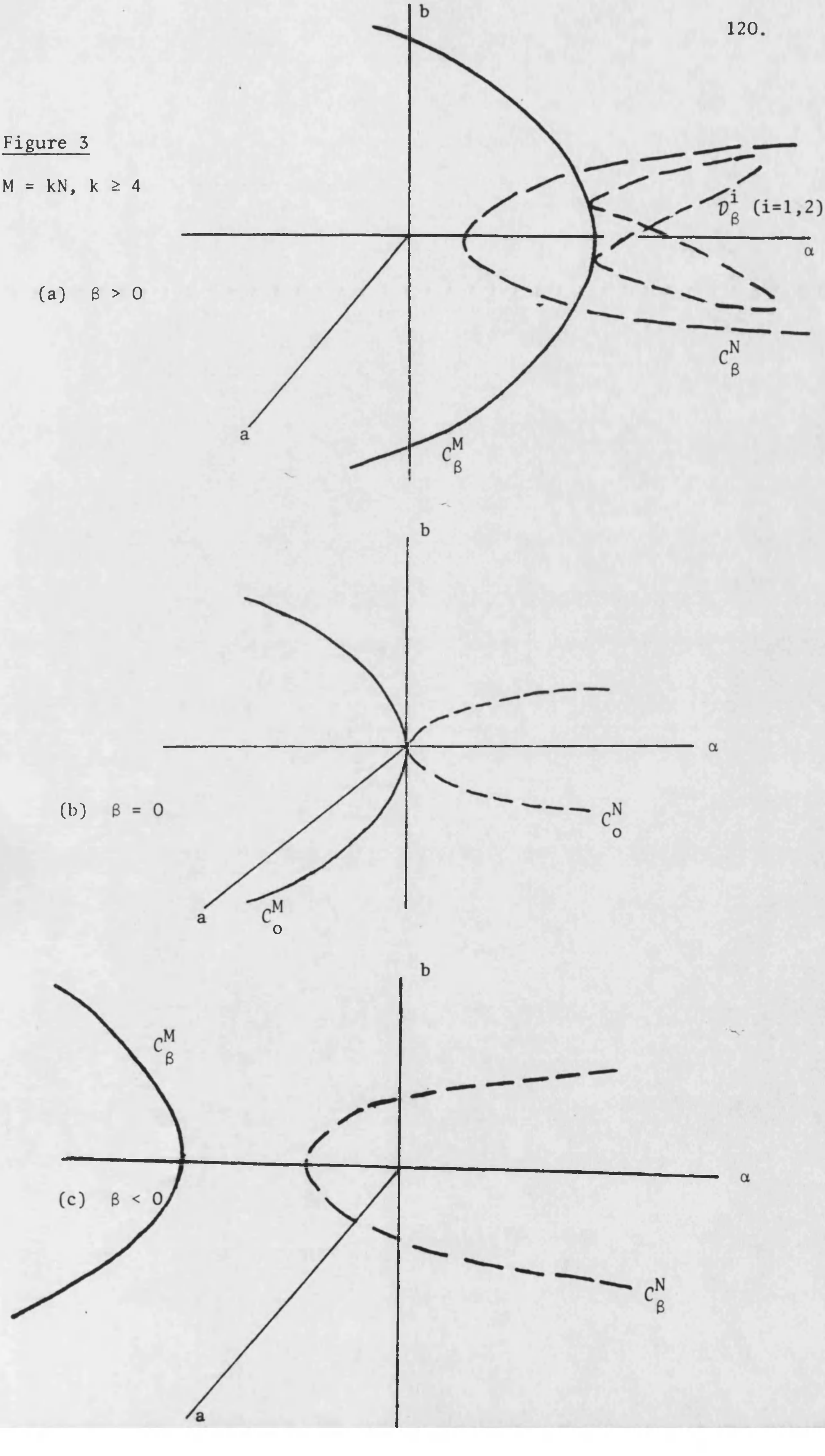
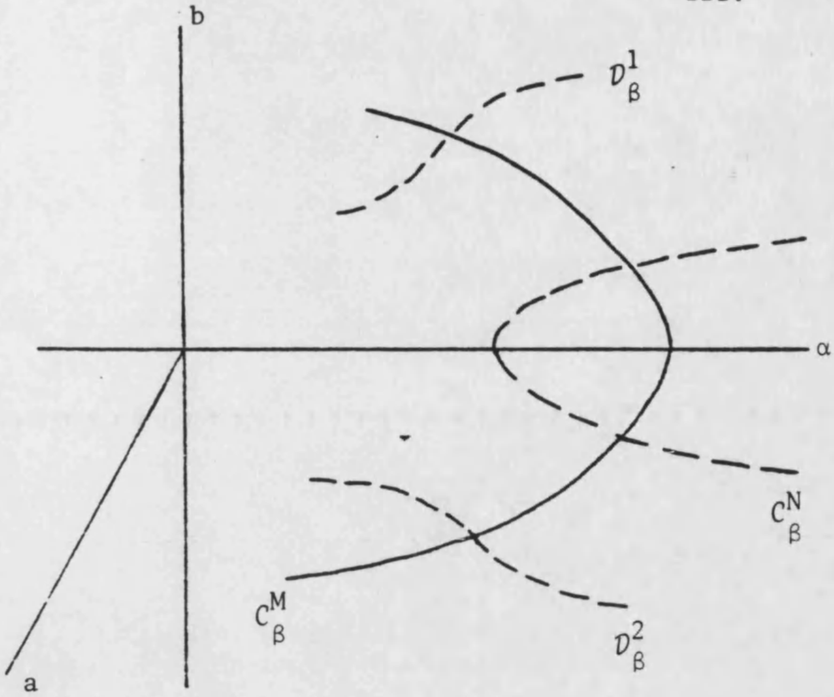


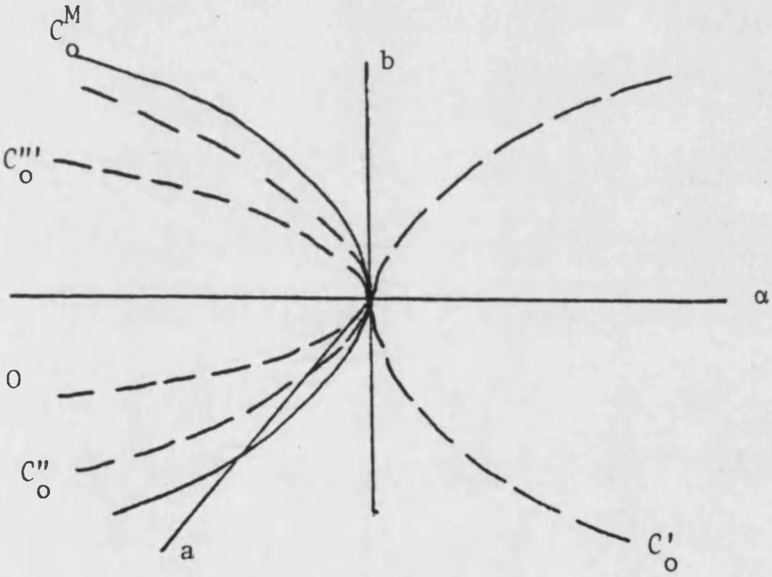
Figure 4

M = 3N

(a) $\beta > 0$



(b) $\beta = 0$



(c) $\beta < 0$

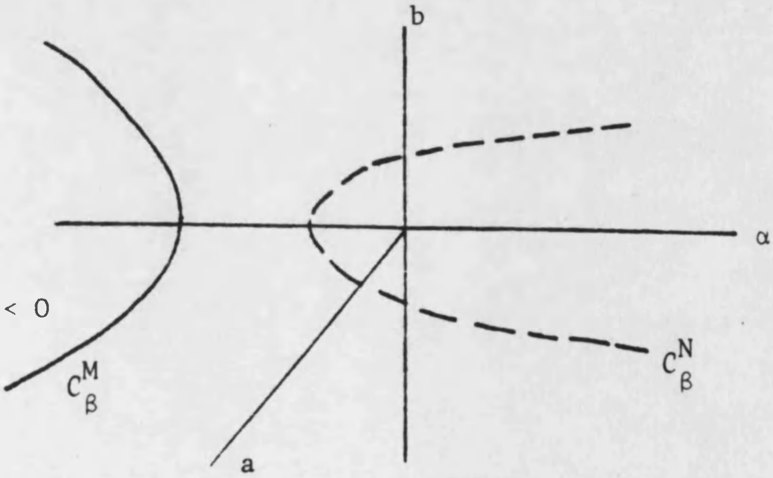
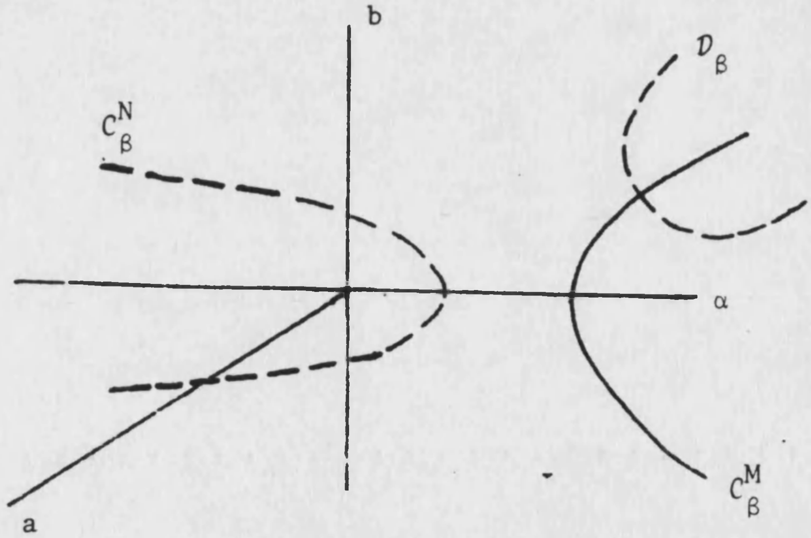


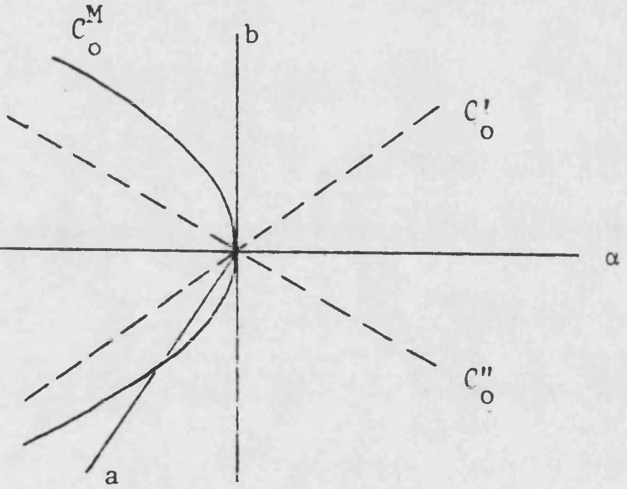
Figure 5

M = 2N

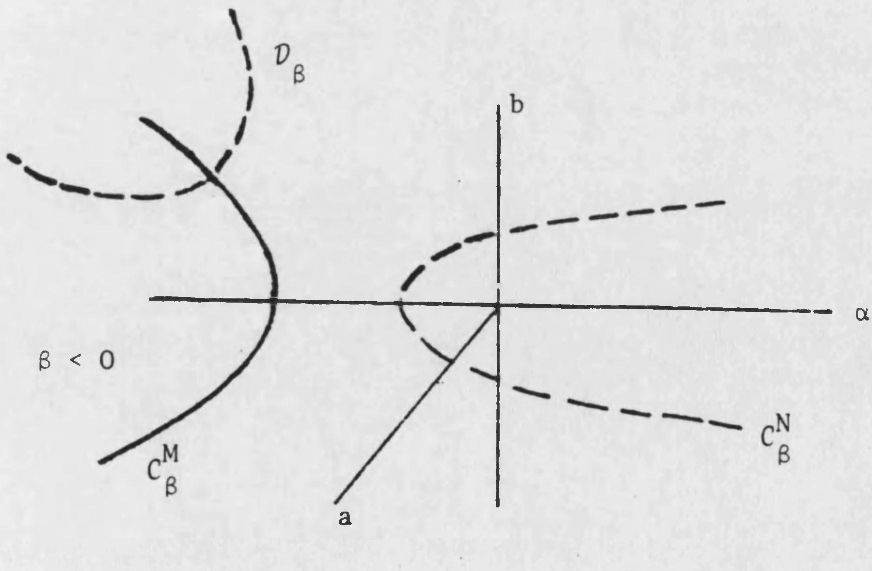
(a) $\beta > 0$



(b) $\beta = 0$



(c) $\beta < 0$



CHAPTER VI

INTERPRETATION OF THE RESULTS

6.1 Introduction

In this chapter we interpret the bifurcation diagrams and the solutions obtained in Chapter V in the context of the underlying hydrodynamical problem. The point is that distinct solutions of (N) may not correspond to distinct capillary-gravity waves. For example if (α,β,θ) is a solution of (N) and $\theta \in X_n$ $(n \geq 1)$, then by Theorem 2.3 $(\alpha,\beta,S_n\theta)$ is also a solution of (N). However by Theorem 2.4, $S_n\theta$ and θ, even though not necessarily equal, correspond to the same capillary-gravity wave. In addition to the question of multiplicity for small-amplitude waves at each (small) value of the parameters α and β, (the perturbations from criticality of the phase speed and surface tension respectively) we address the questions of symmetry and periodicity of these waves. Recall that M and N are fixed natural numbers with $M > N$ and K is their highest common factor. Then, as was discussed in §4.6, the Lyapunov-Schmidt reduction may be carried out in X_K or X_1 with exactly the same bifurcation equations resulting in each case. Hence the greatest minimal period any of these waves can have is $2\pi/K$. However, as we shall see, some have smaller minimal period.

We will say that a wave has <u>crest</u> (or <u>trough</u>) symmetry if the wave has a vertical line of symmetry through a local maximum (or local minimum) of the wave profile. If (α,β,θ) is a solution of (N) then it follows from (2.25) that if $\theta(s_o) = 0$ and $\theta'(s_o) > 0$ $(\theta'(s_o) < 0)$ then the corresponding wave profile has a local maximum (local minimum) corresponding to s_o. We will say that a wave has <u>double symmetry</u> if there are two vertical lines of symmetry whose horizontal separation is not a multiple of the minimal period. In fact because of the formulation of $(\mathcal{N})$ we only consider waves that are $2\pi/K$ periodic and have a vertical line of symmetry. Because of this all waves have double symmetry and the question arises as to whether lines of symmetry go through crests or troughs.

As before the answers to these questions are different according to the various values of M and N. We consider the different cases separately.

6.2 The Case $M \neq kN$

6.2(a) The Primary Curves C_β^N and C_β^M

As we know from §5.3 for all β sufficiently small there are two primary curves, C_β^M and C_β^N which bifurcate from $\alpha = M\beta$ and $\alpha = N\beta$ respectively. C_β^M corresponds to solutions of ($\mathcal{B}$) which have $a = 0$ and it is always a sub-critical pitchfork. It is divided by $\{(M\beta,0)\}$ into two subsets which are denoted ${}^{\pm}C_\beta^M$ according as $\pm b > 0$. $C_\beta^M \subset \mathbb{R}\times X_M$ and hence the corresponding capillary-gravity waves have period $2\pi/M$. The other primary curve C_β^N corresponds to solutions of ($\mathcal{B}$) for which $b = 0$. It is a sub-critical pitchfork if $M < 2N$ and a super-critical pitchfork if $M > 2N$. Its other properties are analogous to those of C_β^M, in particular the corresponding capillary-gravity waves have minimal period $2\pi/N$. Thus the period of the waves along both these primary branches is less than the greatest possible minimal period of the waves arising as solutions to this problem which is $2\pi/K$.

Suppose $(\alpha,\theta) \in C_\beta^N$. Then $\theta \in X_N \backslash X_n$ for any $n > N$. So by Theorems 2.3 and 2.4, $\theta \neq S_N\theta$, but $(\alpha,\beta,S_N\theta)$ is a solution of (N) corresponding to the same capillary-gravity wave. Then since $S_N\theta \in X_N$ it follows from uniqueness that $(\alpha,S_N\theta) \in C_\beta^N$. Hence the two branches ${}^{\pm}C_\beta^N$ correspond, via the pairing $\{(\alpha,\theta), (\alpha,S_N\theta)\}$, to the same set of capillary-gravity waves.

If $(\alpha,\theta) \in C_\beta^N$, then $\theta = a \sin Ns + O(a^2)$, $|a| \ll 1$ (see (5.2)) and so the corresponding wave has either a crest corresponding to $s = 0$ and a trough corresponding to $s = \pi/N$, or vice-versa depending on the sign of a (i.e. depending on whether $(\alpha,\theta) \in {}^{\pm}C_\beta^N$).

All these remarks apply in an analogous way to C_β^M.

The conclusion then is that the solutions on C_β^N and C_β^M have both crest and trough symmetry, and minimal period $2\pi/N$ and $2\pi/M$ respectively.

6.2(b) The Case $N < M < 2N \quad \beta \neq 0$

As we saw in §5.3(a), there are four distinct curves of non-trivial solutions of (N) in a neighbourhood of the origin: C_β^M, C_β^N, $\mathcal{D}_\beta^1$ and $\mathcal{D}_\beta^2$. It remains to examine the latter two. Consider firstly $\beta > 0$. Then according to §5.3(a) $\mathcal{D}_\beta^i$ bifurcates sub-critically from C_β^N and

$$\theta(s) = a \sin Ns + b \sin Ms + O(|(\alpha,\beta)|), \quad (-1)^{i+1}a > 0 \quad \text{if } (\alpha,\theta) \in \mathcal{D}_\beta^i,\ i = 1,2 \quad , \tag{6.1}$$

where

$$(a^2,b^2) = \frac{((a_2r-a_1s)\alpha+(s-r)\beta,\ (a_1q-a_2p)\alpha+(p-q)\beta)}{(ps-qr)} + o(|(\alpha,\beta)|)$$

for $\alpha \le \alpha^*$, where $\alpha^* \approx \dfrac{(p-q)\beta}{a_2p-a_1q}$;

also $(\alpha,\theta) \in {}^{\pm}\mathcal{D}_\beta^i$ according as $\pm b > 0$.

First we observe that if $(\alpha,\theta) \in \mathcal{D}_\beta^1 \cup \mathcal{D}_\beta^2$, then $\theta \in X_K \setminus X_n$ for all $n > K$ and hence the corresponding waves have minimal period $2\pi/K$. Now if $(\alpha,\theta) \in \mathcal{D}_\beta^1 \cup \mathcal{D}_\beta^2$ it follows that $\theta \neq S_K\theta$ but $(\alpha,\beta,S_K\theta)$ is a solution of (N) which corresponds to the same capillary-gravity wave as θ. Hence by uniqueness it follows that $(\alpha,S_K\theta) \in \mathcal{D}_\beta^1 \cup \mathcal{D}_\beta^2$. A calculation using (6.1) then yields that

$$S_K({}^{\pm}\mathcal{D}_\beta^i) \subseteq {}^{\mp}\mathcal{D}_\beta^i \qquad i = 1,2 \qquad \text{if } N/K \text{ is even and } M/K \text{ is odd}$$

$$S_K({}^{\pm}\mathcal{D}_\beta^i) \subseteq {}^{\pm}\mathcal{D}_\beta^j \qquad \{i,j\} = \{1,2\} \qquad \text{if } N/K \text{ is odd and } M/K \text{ is even}$$

$$S_K({}^{\pm}\mathcal{D}_\beta^i) \subseteq {}^{\mp}\mathcal{D}_\beta^j \qquad \{i,j\} = \{1,2\} \qquad \text{if } N/K \text{ and } M/K \text{ are odd.}$$

Hence the four sets ${}^{\pm}\mathcal{D}_\beta^i$, $i = 1,2$ correspond to two distinct sets of waves. We conclude that when $\beta > 0$ the sub-critical branch of capillary-gravity waves to which C_{β}^{N} corresponds contains a point at which a sub-critical secondary pitchfork bifurcation occurs into waves of minimal period $2\pi/K$ and that both branches of the secondary pitchfork correspond to distinct sets of capillary-gravity waves. In a neighbourhood of the secondary bifurcation point the symmetries of the waves on the secondary branches reflect those of the waves on the primary branch from which they bifurcate. Hence it follows from §6.2(a) that in a neighbourhood of the secondary bifurcation point both branches of the fork have double symmetry, crest and trough, if N/K is odd; if N/K is even one branch of the fork has double crest symmetry while the other has double trough symmetry. When $\beta > 0$ there is no secondary bifurcation on C_{β}^{M}.

However when $\beta < 0$ there is no secondary bifurcation along C_{β}^{N} while there is a secondary bifurcation on C_{β}^{M}. The analysis corresponding to the previous case yields that when $\beta < 0$ the sub-critical branch of capillary-gravity waves to which C_{β}^{M} corresponds contains a point where sub-critical secondary pitchfork bifurcation into waves of minimal period $2\pi/K$ occurs. From the formula analogous to (6.1) it follows that in a neighbourhood of the secondary bifurcation point both branches of the fork have waves with double symmetry, crest and trough, if M/K is odd; if M/K is even, one branch of the fork has double crest symmetry, while the other has double trough symmetry. These conclusions are shown in Figure 6.

6.2(c) The Case $N < M < 2N$ $\beta = 0$

In this case the curves $\mathcal{D}_0^1$ and $\mathcal{D}_0^2$ pass through the origin in $\mathbb{R}\times X_K$. According to §5.3(b)

$$\left.\begin{aligned}&\theta(s) = a \sin Ns + b \sin Ms + O(\alpha)\\ &(a^2,b^2) = \frac{((a_2r-a_1s)\alpha,\ (a_1q-a_2p)\alpha)}{(ps-qr)} + o(\alpha)\\ &\alpha \le 0, \quad (-1)^i ab < 0\ ,\end{aligned}\right\} \text{ if } (\alpha,\theta) \in \mathcal{D}_0^i\ . \tag{6.2}$$

(Recall that ${}^{\pm}\mathcal{D}_0^i$ are the subsets of $\mathcal{D}_0^i$ corresponding to $\pm a > 0$.) Then by the uniqueness results of §5.3(b), and a calculation similar to that of §6.2(b), there results that

$S_K({}^{\pm}\mathcal{D}_0^i) \subseteq {}^{\mp}\mathcal{D}_0^j \quad \{i,j\} = \{1,2\}$ if M/K is even and N/K is odd,

$S_K({}^{\pm}\mathcal{D}_0^i) \subseteq {}^{\pm}\mathcal{D}_0^j \quad \{i,j\} = \{1,2\}$ if M/K is odd and N/K is even

while

$S_K({}^{\pm}\mathcal{D}_0^i) \subseteq {}^{\mp}\mathcal{D}_0^i \quad i = 1,2$ if both M/K and N/K are odd.

(Observe that at least one of M/K, N/K is odd since K is the highest common factor of M and N.) Hence, by Theorem 2.4, the four sets ${}^{\pm}\mathcal{D}_0^i$ $(i = 1,2)$ correspond to only two sets of capillary-gravity waves. Therefore there are exactly four distinct branches of capillary-gravity waves bifurcating from $\alpha = 0$ when $\beta = 0$. Clearly the waves on $\mathcal{D}_0^1$ have double symmetry corresponding to the points $s = 0$ and $s = \pi/K$. Whether these will correspond to crest or trough symmetry depends on a calculation of $\theta'(0)$ and $\theta'(\pi/K)$. Now for α sufficiently small, (6.2) gives

$$(Na + Mb)\theta'(0) > 0 \tag{6.3}$$

$$(-1)^{N/K}(Na + (-1)^{(M-N)/K}Mb)\theta'(\pi/K) > 0. \tag{6.4}$$

From now on we must consider the cases in which M/K and N/K are both odd, or one of them is even, separately.

Case I - One of M/K and N/K even

It is clear that at the point corresponding to $s = 0$ the waves on ${}^{+}\mathcal{D}_0^1$ (i.e. $a > 0$, $b > 0$) have crest symmetry and that those on ${}^{-}\mathcal{D}_0^1$ (i.e. $a < 0$, $b < 0$) have trough symmetry.

To investigate the symmetries of the waves at the point corresponding to $s = \pi/K$ note that $\frac{M}{K} - \frac{N}{K}$ is odd so that $(-1)^{(M-N)/K} = -1$ and that it follows from (6.4) that for α sufficiently small

$$\operatorname{sgn}(Na - Mb) = \operatorname{sgn}(Nx_o - My_o) \tag{6.5}$$

where (x_o, y_o) is the point of intersection in the first quadrant of the ellipses

$$px^2 + ry^2 = a_1$$

$$qx^2 + sy^2 = a_2 \quad .$$

Now

$$0 < \left\{ \left(\frac{p}{a_1} - \frac{q}{a_2} \right) \Big/ \left(\frac{s}{a_2} - \frac{r}{a_1} \right) \right\}^{1/2} = \frac{y_o}{x_o} \tag{6.6}$$

and so it is clear that

$$\operatorname{sgn}(Na - Mb) = \operatorname{sgn} \left\{ N^2 \left| \frac{s}{a_2} - \frac{r}{a_1} \right| - M^2 \left| \frac{p}{a_1} - \frac{q}{a_2} \right| \right\} \quad . \tag{6.7}$$

Then (6.7) coupled with (6.4) enables us to determine the sign of $\theta'(\pi/K)$.

We therefore wish to evaluate the bracketed term in (6.7). Recall that $N > 1$, $N < M < 2N$ and precisely one of M/K and N/K is even. For reasons which will become clear shortly it is convenient to write $M = N+d$ where $1 \le d \le N-1$.

Now a calculation yields that

$$N^2\left|\frac{s}{a_2} - \frac{r}{a_1}\right| - (N+d)^2\left|\frac{p}{a_1} - \frac{q}{a_2}\right|$$

$$= \frac{d(N^5 - 45N^4d - 17N^3d^2 + 77N^2d^3 + 72Nd^4 + 20d^5)}{8N(N+d)^2(N-d)(N+2d)} \quad . \tag{6.8}$$

It is then clear that the denominator is always positive and so we wish to determine the sign of

$$S(d) = N^5 - 45N^4d - 17N^3d^2 + 77N^2d^3 + 72Nd^4 + 20d^5 \tag{6.9}$$

for fixed $N > 1$ and $1 \le d \le N-1$.

Now,

$$S(1) = N^5 - 45N^4 - 17N^3 + 77N^2 + 72N + 20$$

and a calculation shows that this is negative for $2 \le N \le 45$ and positive for $N > 45$. The largest value d can take is $N-1$ and

$$S(N-1) = 108N^5 - 540N^4 + 846N^3 - 565N^2 + 172N - 20$$

and an elementary calculation shows that this is positive for all $N \ge 2$. Therefore as d runs through its possible values $S(d)$ certainly changes sign for $2 \le N \le 45$. However it in fact changes sign for $N > 45$ also. For if N is even then $N/2$ is an integer whereas if N is odd, $\frac{N+1}{2}$ is an integer and

$$S(N/2) = -11N^5$$

$$S(\frac{N+1}{2}) = -11N^5 - 19N^4 + 57.876N^3 - 33.875N^2 + 7.625N - 0.625$$

both of which quantities are negative for $N \geq 2$.

The point of these calculations is to show that there does not seem to be any simple condition on M and N which determines $\mathrm{sgn}(Na - Mb)$, this depends on the actual numbers chosen. For example if $N = 3$ and $M = 4$ (so $d = 1$), $S(1) < 0$, therefore $\mathrm{sgn}(Na - Mb) < 0$ and the waves on ${}^{+}\mathcal{D}_0^1$ have a crest corresponding to $s = \pi/K$; while if $N = 7$ and $M = 12$ (so $d = 5$), $S(5) > 0$, therefore $\mathrm{sgn}(Na - Mb) > 0$ and the waves on ${}^{+}\mathcal{D}_0^1$ have a trough corresponding to $s = \pi/K$.

The symmetries of the waves on $\mathcal{D}_0^i$, $i = 1,2$ when $M < 2N$ and exactly one of M/K and N/K is even may then be described as follows: For $\beta = 0$, in addition to C_0^M and C_0^N, two distinct branches of capillary-gravity waves bifurcate sub-critically from $\alpha = 0$. Either the waves on one branch have double crest symmetry while those on the other have double trough symmetry or both have crest and trough symmetry. The former possibility occurs if $(-1)^{N/K}\mathrm{sgn}(Na - Mb)$ is positive and the latter if it is negative. Of particular physical interest is the case $M = N+1$. This means (see §4.4) that the smallest element of the spectrum of the linearised operator (corresponding physically to the slowest phase speed at which a bifurcation into a non-trivial flow may take place) is a double eigenvalue. The symmetries of the waves on $\mathcal{D}_0^i$ ($i = 1,2$) may easily be obtained in this case from (6.9) by putting $d = 1$ and it is then an elementary calculation to show that the resulting expression is positive for $N > 45$ and negative for $N \leq 45$. It is thus clear (from (6.4)) that in this case, there is a simple criterion, depending on the size of N, for determining the symmetries of the waves. The results are as follows: For $\beta = 0$ and $N \leq 45$ even or $N > 45$ odd, in addition to C_0^N and C_0^{N+1}, two distinct waves with double symmetry, both crest and trough, bifurcate sub-critically at $\alpha = 0$. For $\beta = 0$ and $N \leq 45$ odd or $N > 45$ even, in addition to C_0^N and C_0^{N+1} two distinct waves with double symmetry, one double crest and one double trough, bifurcate sub-critically at $\alpha = 0$.

Remark. It is a consequence of the formulation of the problem as an integral equation and the choice of spaces in the Lyapunov-Schmidt procedure that all solutions θ correspond to waves with period $2\pi/K$ which enjoy double symmetry. However, the conclusions that the symmetries of the waves on the secondary branches can further be identified as crest-crest, trough-trough or crest-trough depends on a local argument in a neighbourhood of the bifurcation point. As we have just seen, for M and N fixed, the symmetries of the waves on the secondary branches may change as β moves from a positive to a negative value. Hence when $\beta \neq 0$, in a small neighbourhood of the secondary bifurcation point the waves on the secondary branches reflect the symmetries of the waves from which they bifurcate, whereas outside this neighbourhood, by continuity, they reflect the symmetries of the solutions of minimal period $2\pi/K$ that bifurcate from $(\alpha,\theta) = (0,0)$ when $\beta = 0$. It is a consequence of the change of symmetries that this small neighbourhood shrinks to zero in $\mathbb{R}\times X_K$ as β $(\neq 0) \to 0$. What is most surprising is how this symmetry depends on the sizes of M and N.

<u>Case II - Both M/K and N/K odd</u>

Now the difference $\frac{M}{K} - \frac{N}{K}$ is even. It is therefore clear from (6.3) and (6.4) that the waves on ${}^{+}\mathcal{D}_0^1$ (i.e. a > 0, b > 0) have crest symmetry corresponding to s = 0 and trough symmetry corresponding to s = π/K. To determine the symmetries of the waves on ${}^{+}\mathcal{D}_0^2$ (i.e. a > 0, b < 0) we must, as before, examine

$$\operatorname{sgn}(Na - Mb) = \operatorname{sgn}(Nx_o - My_o) \tag{6.10}$$

and calculations similar to those just performed show that this may be positive or negative depending on the values of M and N. However by (6.3) and (6.4) this fact is of little relevance since no matter what the sign of (6.10) is, the waves on ${}^{+}\mathcal{D}_0^2$ exhibit opposite characteristics at s = 0 and s = π/K (i.e. one point corresponds to a trough and the other to a crest). Hence when N < M < 2N and both M/K and N/K are odd, in addition to C_0^M and C_0^N there is sub-critical bifurcation of two distinct wave branches each of minimal period $2\pi/K$ from $\alpha = 0$. Both waves enjoy both crest and trough symmetry. In this case the symmetries of the waves on the secondary branches do not change type as β moves from a positive to a negative value with M and N held fixed. These conclusions are pictured in Figure 6 . Note the same figure suffices for both Case I and Case II because the only difference between these cases is in the symmetry of the waves and not in their multiplicity or period.

<u>6.2(d) The Case M > 2N</u>

As we remarked in §6.2(a), the primary curves C_β^N and C_β^M are present for all values of β. It follows from §5.3(a) that for $\beta > 0$ there are four secondary bifurcation points, two on each of C_β^N and C_β^M , and that in addition to the primary solution curves there are the secondary curves $\mathcal{D}_\beta^1$ and $\mathcal{D}_\beta^2$. If $(\alpha,\theta) \in \mathcal{D}_\beta^i$ $(i = 1,2)$, then by §5.3(a)

$$\theta(s) = a \sin Ns + b \sin Ms + O(|(\alpha,\beta)|)$$

where

$$(a^2,b^2) = \frac{((a_2r-a_1s)\alpha+(s-r)\beta,\ (a_1q-a_2p)\alpha+(p-q)\beta)}{(ps-qr)} + o(|(\alpha,\beta)|)$$

and $\qquad$ (6.11)

$$\alpha_* \le \alpha \le \alpha^*$$

where

$$\alpha^* \approx \frac{(p-q)\beta}{a_2p-a_1q}\ , \qquad \alpha_* \approx \frac{(s-r)\beta}{a_1s-a_2r}\ .$$

(Recall that along ${}^\pm\mathcal{D}_\beta^1$, $a > 0$ and $\pm b > 0$, while along ${}^\pm\mathcal{D}_\beta^2$, $a < 0$ and $\pm b > 0$.) Thus it follows from the same reasoning as in the previous case that if $(\alpha,\theta) \in \mathcal{D}_\beta^1 \cup \mathcal{D}_\beta^2$ then $(\alpha, S_K\theta) \in \mathcal{D}_\beta^1 \cup \mathcal{D}_\beta^2$ and $\theta \neq S_K\theta$ but θ and $S_K\theta$ correspond to the same capillary-gravity wave. A calculation based on (6.11) then yields that

$S_K({}^\pm\mathcal{D}_\beta^i) \subseteq {}^\pm\mathcal{D}_\beta^j \qquad \{i,j\} = \{1,2\} \quad$ if M/K is even and N/K is odd,

$S_K({}^\pm\mathcal{D}_\beta^i) \subseteq {}^\mp\mathcal{D}_\beta^i \qquad i = 1,2 \qquad\quad$ if M/K is odd and N/K is even,

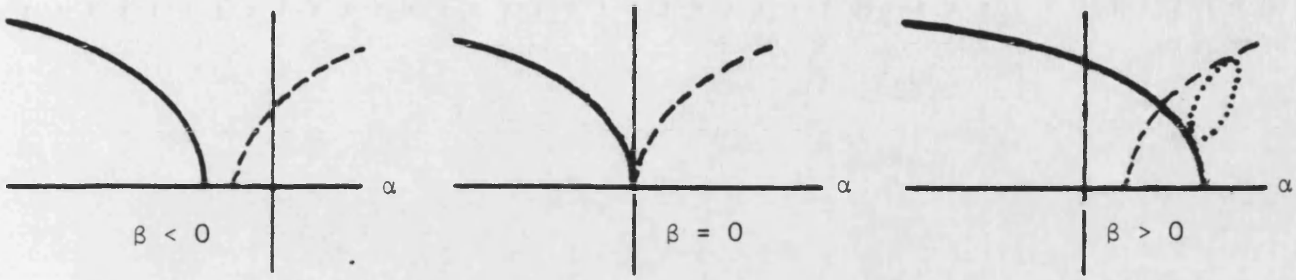
$S_K({}^\pm\mathcal{D}_\beta^i) \subseteq {}^\mp\mathcal{D}_\beta^j \qquad \{i,j\} = \{1,2\} \quad$ if M/K and N/K are odd.

Hence the four sets ${}^\pm\mathcal{D}_\beta^i$ $(i = 1,2)$ correspond to only two distinct sets of waves. We conclude that when $\beta > 0$ both primary solution branches of capillary-gravity waves contain a point at which a secondary pitchfork bifurcation occurs into waves of minimal period $2\pi/K$. This secondary bifurcation is sub-critical from the branch corresponding to C_β^N and super-critical from that corresponding to C_β^M. Moreover it follows from (6.11) that in a neighbourhood of the secondary bifurcation point on the branch corresponding to C_β^N the waves on both branches of the secondary pitchfork enjoy both crest and trough symmetry if N/K is odd; if N/K is even then in this neighbourhood one branch of the fork has double crest symmetry, the other double trough. Similarly in a neighbourhood of the other secondary bifurcation point the waves on the two branches of the pitchfork enjoy crest and trough symmetry if M/K is odd; while if M/K is even one branch has double crest symmetry and the other double trough.

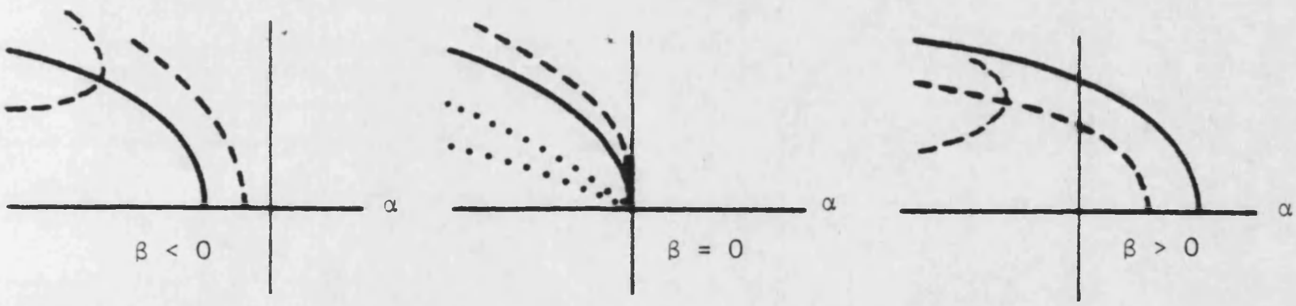
If $\beta \leq 0$ the only known solution curves present are the primary curves corresponding to C_β^N and C_β^M. These conclusions are shown in Figure 6.

Figure 6

$M \neq kN$



(a) $M > 2N$



(b) $N < M < 2N$

This figure, and also figure 7, are intended to show the number of distinct capillary-gravity waves corresponding to various values of the phase speed. They are of a qualitative nature only. The abscissa α is the perturbation of the phase speed squared from criticality whereas the ordinate is a measure of the wave amplitude.

6.3 The Case M = kN

(a) $k \geq 4$

This is similar to the case $M \neq kN$ and $M > 2N$ with the additional complications discussed in §§4.6, 6.9 and 5.4 arising from the lack of symmetry in the problem and the consequent differences in the higher order terms of the bifurcation equations.

For all values of β there are two primary curves. One, C_β^M, is a sub-critical pitchfork bifurcation from $\alpha = M\beta$. Both branches of the pitchfork correspond to the same set of capillary-gravity waves. These waves have minimal period $2\pi/M$ and enjoy both crest and trough symmetry. The other, $\tilde{C}_\beta^N$, is a super-critical pitchfork from $\alpha = N\beta$, both branches corresponding to the same set of waves whose minimal period is $2\pi/N$ and which enjoy both crest and trough symmetry. Note that in this situation the greatest minimal period of any wave is $2\pi/N$ (since the highest common factor of M and N is N) and the waves corresponding to $\tilde{C}_\beta^N$ have no smaller period. This is in contrast to the situation when $M \neq kN$.

If $\beta > 0$, there is a single secondary bifurcation point on the primary branch of waves which bifurcates from $\alpha = M\beta$. At this point a super-critical pitchfork bifurcation into waves of minimal period $2\pi/N$ occurs. The two branches of the pitchfork consist of distinct sets of capillary-gravity waves. In the neighbourhood of the secondary bifurcation point the symmetries of these waves reflect the symmetries of the waves on the primary branch: thus if M/K is odd they have both crest and trough symmetry while if M/K is even one has double crest symmetry, the other double trough symmetry. As remarked in §5.4 it is not known whether this secondary curve intersects the other primary branch, but it seemsunlikely.

If $\beta \leq 0$ only the primary branches are present. See Figure 7.

6.3(b) The Case M = 3N

For all values of $\beta \neq 0$ there are the two primary curves: the super-critical pitchfork $\tilde{C}_\beta^N$ bifurcating from $\alpha = N\beta$, and the sub-critical pitchfork C_β^{3N} bifurcating from $\alpha = 3N\beta$. By the usual symmetry considerations each branch of a fork corresponds to the same set of capillary-gravity waves. The waves corresponding to C_β^{3N} have minimal period $2\pi/3N$ while those corresponding to $\tilde{C}_\beta^N$ have minimal period $2\pi/N$. Both sets of waves enjoy crest and trough symmetry.

If $\beta > 0$ there are two secondary curves, $\mathcal{D}_\beta^1$ and $\mathcal{D}_\beta^2$, which bifurcate transcritically from ${}^+C_\beta^{3N}$ and ${}^-C_\beta^{3N}$ respectively. Recall that ${}^\pm\mathcal{D}_\beta^i$ $(i = 1,2)$ denotes the subsets of these curves along which $\pm a > 0$. It follows from arguments analogous to those used in §6.2 that

$$S_N({}^\pm\mathcal{D}_\beta^i) \subset {}^\mp\mathcal{D}_\beta^j \qquad \{i,j\} = \{1,2\}$$

and so the four solution curves ${}^\pm\mathcal{D}_\beta^i$ $(i = 1,2)$ correspond to only two distinct sets of capillary-gravity waves. In physical terms then, we conclude that when $\beta > 0$ the primary branch of waves with minimal period $2\pi/3N$ contains a point at which a transcritical secondary bifurcation into waves of minimal period $2\pi/N$ occurs. The two subsets of the secondary branch correspond to different families of capillary-gravity waves. In a neighbourhood of the secondary bifurcation point all the waves on the secondary branch enjoy both crest and trough symmetry.

If $\beta < 0$ only the primary solution branches C_β^{3N} and $\tilde{C}_\beta^N$ are present.

If $\beta = 0$ there is multiple bifurcation of four pitchfork curves from $\alpha = 0$. One is C_0^{3N} and its properties are by now familiar. The other three which (see §5.4) are denoted C_0', C_0'', C_0''' consist of two sub-critical pitchforks and one super-critical pitchfork. According to §5.4, if (α,θ) lies on any of these curves then

$$\left.\begin{aligned} \theta(s) &= a \sin Ns + aw_i \sin 3Ns + O(a^2) \\ \alpha &= \frac{a^2}{4N}\left(\frac{77}{6} + \frac{41}{2} w_i + \frac{23}{9} w_i^2\right) + O(a^3) \end{aligned}\right\}, \qquad (6.12)$$

for $|a| << 1$

where $w_1 \sim -5.3$, $w_2 \sim -1.3$, $w_3 \sim 0.83$, each value of w_i corresponding to a different solution curve. A calculation using (6.12) now shows that both branches of a pitchfork correspond to the same set of capillary-gravity waves and that these waves have minimal period $2\pi/N$ and enjoy both crest and trough symmetry.

In physical terms then: when $\beta = 0$, four distinct branches of capillary-gravity waves bifurcate from $\alpha = 0$. Three of the bifurcations are sub-critical and one is super-critical. One of the sub-critical branches consists of waves whose minimal period is $2\pi/3N$ while the waves on all the other branches have period $2\pi/N$. All these waves exhibit both crest and trough symmetry. See Figure 7.

6.3(c) The Case M = 2N

If $\beta \neq 0$ there are two primary bifurcation curves. One, C_β^{2N}, is a sub-critical pitchfork bifurcation from $\alpha = 2N\beta$. The other, $\tilde{C}_\beta^{N}$, bifurcates from $\alpha = N\beta$, it is a sub-critical pitchfork if $\beta > 0$ and a super-critical pitchfork if $\beta < 0$. As usual both branches of a pitchfork correspond to the same set of capillary-gravity waves. Those corresponding to C_β^{2N} have minimal period π/N while those corresponding to $\mathcal{C}_\beta^N$ have minimal period $2\pi/N$. Both sets of waves exhibit both crest and trough symmetry. In addition C_β^{2N} contains a secondary bifurcation point which lies on ${}^+C_\beta^{2N}$ if $\beta > 0$ and ${}^-C_\beta^{2N}$ if $\beta < 0$. The secondary curve $\mathcal{D}_\beta$ is a super(sub-) critical bifurcation if $\beta > 0$ ($\beta < 0$) and both branches of the fork correspond to the same set of capillary-gravity waves whose minimal period is $2\pi/N$. In a neighbourhood of the secondary bifurcation point the waves on the secondary branch reflect the symmetries of the waves on the branch from which they bifurcate. Hence they exhibit double crest symmetry if $\beta > 0$ and double trough symmetry if $\beta < 0$. See Figure 7.

When $\beta = 0$ there is primary bifurcation of three non-trivial solution curves through the origin. One is C_0^{2N} whose properties are by now familiar. The other curves consist of two transcritical bifurcations from the origin which we have denoted C_0' and C_0''. According to §5.4 if $(\alpha,\theta) \in C_0' \cup C_0''$ then

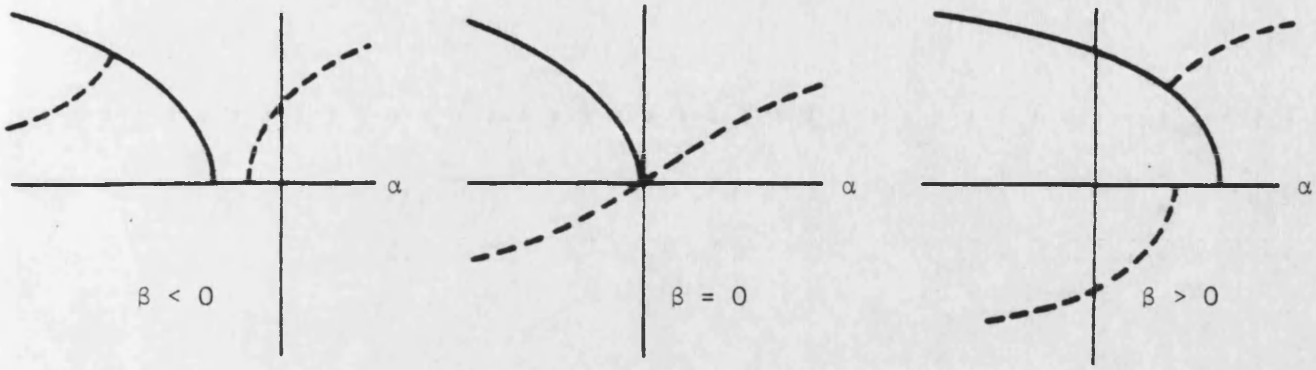
$$\theta(s) = \pm\frac{4N\alpha}{3}\sin Ns + \frac{4N\alpha}{3}\sin 2Ns + o(\alpha), \quad |\alpha| << 1. \tag{6.13}$$

Hence for a given value of α it is easily seen that both $(\alpha,\theta) \in C_0' \cup C_0''$ and $(\alpha,S_N\theta) \in C_0' \cup C_0''$. Moreover both these solutions correspond to the same water-wave. Further these waves have minimal period $2\pi/N$ and it is an easy calculation to discover that those corresponding to positive values of α enjoy double crest symmetry while those corresponding to negative α have double trough symmetry.

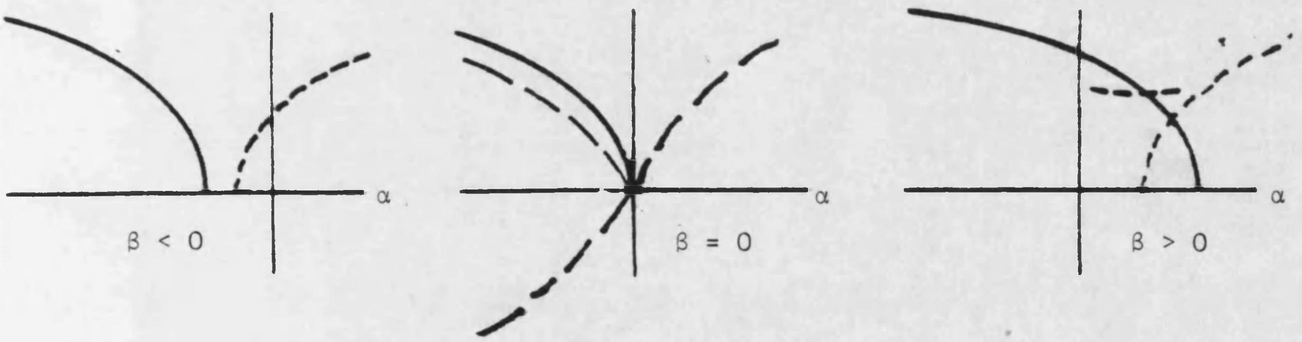
In physical terms then, when $\beta = 0$, the bifurcation of capillary-gravity waves consists of three primary curves bifurcating from the origin. One is a sub-critical branch of π/N-periodic waves, symmetric about crest and troughs. The other two consist of one super- and one sub-critical curve each consisting of $2\pi/N$-periodic waves. Those on the super-critical branch enjoy double crest symmetry, while those on the sub-critical branch enjoy double trough symmetry. See Figure 7.

Figure 7

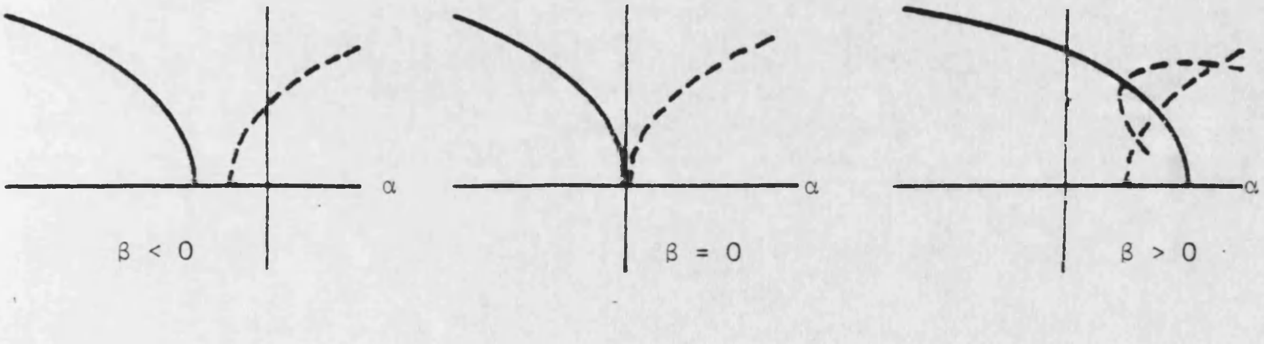
M = kN



(a) M = 2N



(b) M = 3N



(c) M ≥ 4N

CONCLUSIONS

An extensive study has been made of the capillary-gravity wave problem, particularly the local aspects of this problem. Various symmetries inherent in the physical situation have been identified and all the bifurcation points which arise as a result of these symmetries have been located. Two parameters ν and γ, which are measures of the magnitude of the phase speed and the surface tension respectively, have been defined and it has been shown that for any fixed value of γ there is a countable set of values of ν, known as eigenvalues, at which bifurcation of small amplitude capillary-gravity waves from the flat free surface can occur. At a particular eigenvalue there may be only one family of capillary-gravity waves which can bifurcate from the flat free surface or there may be two, three or four distinct wave families which differ from each other either in their symmetry or in their minimal periods. Thus for given values of the parameters ν and γ, more than one distinct wave is possible. For instance if $\gamma \sim (MN)^{-1}$, $M > 2N$ and $M \neq kN$, then for certain values of ν greater than, but close to, $M^{-1} + N^{-1}$, four distinct water-waves can occur (see Figure 6). It has also been shown that perturbations of the surface tension may have a dramatic effect on the number of solutions. For instance if $\gamma = (MN)^{-1}$ then for all values of $\nu \sim M^{-1}+N^{-1}$ only one water-wave is possible (see Figure 6). These waves are all part of a global continuum of solutions which must satisfy one of the alternatives enumerated in Chapter IV.

There are a number of further questions regarding this problem. Firstly, in the situation where a number of different waves are possible, we have not attempted to decide, on physical criteria, which are the more likely to occur; secondly we have not considered the question of stability of the wave profile. Neither have we attempted to discover which of the different possibilities described in Chapter IV are actually satisfied by the global continuum of solutions. Finally it would be very interesting to determine "a priori" bounds on the angle θ which the wave profile makes with the horizontal, analogous to those obtained by Amick & Toland (1981) and Amick (to appear) for the corresponding waves in the absence of surface tension.

Throughout this thesis the depth of the channel has always been assumed infinite. An obvious generalisation of the work presented here is the case when the depth is finite. It is possible to derive an equation analogous to ($\mathcal{N}$) in that case, the major difference being that τ is a non-linear function of θ. (This equation is of independent interest since obtaining it is the first step in the study of the solitary capillary-gravity wave problem.) An analysis of the equation on similar lines yields the conclusion that in the case of finite depth the results are essentially the same as those for infinite depth. These findings will be given elsewhere.

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